

GEOMETRY OF 2-STEP NILPOTENT LIE GROUPS

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Introduction A Lie algebra \mathfrak{N} is said to be *2-step nilpotent* if its commutator ideal $[\mathfrak{N}, \mathfrak{N}]$ is nontrivial and contained in the center of \mathfrak{N} . Such a Lie algebra \mathfrak{N} is said to be of *type (p,q)* if the ideal $[\mathfrak{N}, \mathfrak{N}]$ has dimension p and codimension q in \mathfrak{N} . An inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{N} determines uniquely a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on the simply connected Lie group N with Lie algebra \mathfrak{N} . The algebraic structure of $\mathfrak{N}, \langle \cdot, \cdot \rangle$ encodes the dynamical and geometric structure of $\{N, \langle \cdot, \cdot \rangle\}$ and the quotients of N by discrete subgroups Γ .

In this article we describe some of the structure of a standard, metric, 2-step nilpotent Lie algebra $\{\mathfrak{N}, \langle \cdot, \cdot \rangle\}$ of type (p,q) . As a vector space \mathfrak{N} is the direct sum $\mathbb{R}^q \oplus W$, where W is a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$, the Lie algebra of $q \times q$ real, skew symmetric matrices. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{R}^q \oplus W$ that keeps the factors orthogonal, agrees with the standard inner product on \mathbb{R}^q and on W is given by $\langle Z, Z' \rangle = -\text{trace}(ZZ')$. The 2-step nilpotent Lie bracket on $\mathfrak{N} = \mathbb{R}^q \oplus W$ is defined by the conditions that W lie in the center of \mathfrak{N} and $\langle [v, w], Z \rangle = \langle Z(v), w \rangle$ for all $v, w \in \mathbb{R}^q$ and $Z \in W$. It is easy to see that $[\mathfrak{N}, \mathfrak{N}] = W$. Moreover W is the center of $\mathfrak{N} \Leftrightarrow W$ has trivial kernel in \mathbb{R}^q . The most interesting cases are the nongeneric ones where W has some extra structure such as being a subalgebra of $\mathfrak{so}(q, \mathbb{R})$ or, more generally, a Lie triple system in $\mathfrak{so}(q, \mathbb{R})$. See section 1 for a discussion of Lie triple systems.

Every 2-step nilpotent Lie algebra \mathfrak{N} of type (p,q) is isomorphic to one of these standard metric models, so we may think of the inner product $\langle \cdot, \cdot \rangle$ defined above as a preferred inner product on \mathfrak{N} . Any other inner product $\langle \cdot, \cdot \rangle^*$ on \mathfrak{N} can be obtained from $\langle \cdot, \cdot \rangle$, up to isometry, by changing $\langle \cdot, \cdot \rangle$ only on W , keeping the factors \mathbb{R}^q and W orthogonal and preserving the standard inner product on \mathbb{R}^q .

We briefly describe the contents of the article. An expanded version of the article may be found on the author's website.

After some preliminary information in the first section we address in section 2 the problem of finding first integrals for the geodesic flow on a connected Lie group H with a left invariant metric $\langle \cdot, \cdot \rangle^*$. The inner product $\langle \cdot, \cdot \rangle^*$ on the Lie algebra \mathfrak{h} and its extension to a left invariant metric $\langle \cdot, \cdot \rangle^*$ on H define Poisson structures on TH and \mathfrak{h} . We review briefly the definition and properties of

these Poisson structures and discuss the Gauss map $G : TH \rightarrow \mathfrak{h}$, which is an anti Poisson map. If $\mathbf{E} : \mathfrak{h} \rightarrow \mathbb{R}$ is the energy function given by $\mathbf{E}(X) = (1/2) \langle X, X \rangle^*$, then the first integrals for the Hamiltonian vector field $X_{\mathbf{E}}$ on \mathfrak{h} correspond one-one to left invariant first integrals $f : TH \rightarrow \mathbb{R}$ for the geodesic flow on TH. In addition each element A of \mathfrak{h} defines a first integral $f_A : TH \rightarrow \mathbb{R}$ for the geodesic flow on TH, but this first integral f_A is not a left invariant function in general unless A lies in the center of \mathfrak{h} . At the end of section 2 we describe some first integrals for the geodesic flow on TN that are particular to the case that N is simply connected and 2-step nilpotent. The Ricci tensor of $\{N, \langle \cdot, \cdot \rangle^*\}$ plays an important role in this part.

In section 3 we study the automorphisms and derivations of a 2-step nilpotent lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$ of type (p, q) , and we derive canonical descriptions of $\text{Aut}(\mathfrak{N})$ and $\text{Der}(\mathfrak{N})$. An important feature in these descriptions is the stability group $\overline{G}_W = \{g \in GL(q, \mathbb{R}) : g(W) = W\}$ and its Lie algebra $\overline{\mathfrak{G}}_W$, where an element g of $GL(q, \mathbb{R})$ acts on $\mathfrak{so}(q, \mathbb{R})$ by $g(Z) = gZg^t$ for all $Z \in \mathfrak{so}(q, \mathbb{R})$. At the end of this section we describe some recent work of J. Lauret on the existence of a new family of Anosov diffeomorphisms for compact nilmanifolds. The existence of these is equivalent to the algebraic problem of finding an automorphism ϕ of a nilpotent Lie algebra \mathfrak{N} that preserves a \mathbb{Z} structure on \mathfrak{N} , has determinant ± 1 and has eigenvalues with modulus different from 1. We present some additional nonexistence results for 2-step nilpotent Lie algebras $\mathfrak{N} = \mathbb{R}^q \oplus W$ in the cases that $\overline{G}_W = \mathbb{R}$ (almost always true for $p \geq 3$), $W = \mathfrak{so}(q, \mathbb{R})$ and q is even or W is defined by a representation on \mathbb{R}^q of the real, negative definite Clifford algebra $\mathcal{C}\ell(p)$.

In section 4 we describe the moduli space $\mathfrak{N}(p, q)$ of isomorphism classes of 2-step nilpotent Lie algebras of type (p, q) . The main result, proved in [E4], is that $\mathfrak{N}(p, q)$ may be identified bijectively with the orbit space $G(p, \mathfrak{so}(q, \mathbb{R})) / GL(q, \mathbb{R})$, where $G(p, \mathfrak{so}(q, \mathbb{R}))$ denotes the Grassmann manifold of p -dimensional subspaces of $\mathfrak{so}(q, \mathbb{R})$. If $\mathfrak{N} = \mathbb{R}^q \oplus W$ is 2-step nilpotent of type (p, q) , then its equivalence class $[\mathfrak{N}]$ is the orbit $GL(q, \mathbb{R})(W)$ in $\mathfrak{so}(q, \mathbb{R})$. The Lie algebra $\overline{\mathfrak{G}}_W$ of the isotropy group $\overline{G}_W = \{g \in GL(q, \mathbb{R}) : g(W) = W\}$ is generically 1-dimensional except in a small number of cases that we write down explicitly. In [E7] we compute $\overline{\mathfrak{G}}_W$ in the nongeneric cases that $W = \mathfrak{so}(3, \mathbb{R}) \subseteq \mathfrak{so}(q, \mathbb{R})$ and q is even or W is defined by a representation of the Clifford algebra $\mathcal{C}\ell(p)$ on \mathbb{R}^q .

In section 5 we describe the space of equivalence classes of *rational structures* on a 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$ of type (p, q) . A rational structure on \mathfrak{N} is a \mathbb{Q} -Lie algebra $\mathfrak{N}_{\mathbb{Q}} = \mathbb{Q} \text{-span}(\mathfrak{B})$, where \mathfrak{B} is a basis of \mathfrak{N} whose structure constants are rational. We say that two rational structures $\mathbb{Q} \text{-span}(\mathfrak{B}_1)$ and $\mathbb{Q} \text{-span}(\mathfrak{B}_2)$ are *equivalent* if there is an automorphism of \mathfrak{N} that carries one to the other. By a result of Mal'cev there is a one one correspondence between rational structures on \mathfrak{N} and commensurability classes of lattices in N , the simply connected nilpotent Lie group with Lie algebra \mathfrak{N} .

In order to have at least one rational structure on $\mathfrak{N} = \mathbb{R}^q \oplus W$ we may assume, without loss of generality, that W is a *standard rational subspace* of $\mathfrak{so}(q, \mathbb{R})$; that is W has a basis whose elements have rational entries. Clearly the set of standard rational subspaces of $\mathfrak{so}(q, \mathbb{R})$ is invariant under the action of $GL(q, \mathbb{Q})$. Now let $\overline{G}_{\mathbb{Q}}(W)$ denote the set of all standard rational p -dimensional subspaces of $\mathfrak{so}(q, \mathbb{R})$ that lie in the orbit $\overline{G}(W)$, where $\overline{G} = GL(q, \mathbb{R})$. In section 5.5 we show that the space of equivalence classes of rational structures on $\mathfrak{N} = \mathbb{R}^q \oplus W$ may be identified with the orbit space $\overline{G}_{\mathbb{Q}}(W) / GL(q, \mathbb{Q})$. To understand properly the nature of this orbit space would seem to require, once again, an understanding of the isotropy Lie algebra $\overline{\mathfrak{G}}_W$ and the isotropy group G_W .

In section 6 we consider some criteria for the isotropy Lie algebra $\overline{\mathfrak{G}}_W$ to be self adjoint, and some consequences of self adjointness that are needed for the discussion of Anosov diffeomorphisms in section 3. In particular if $\mathfrak{N} = \mathbb{R}^q \oplus W$ is 2-step nilpotent of type (p, q) , then $\overline{\mathfrak{G}}_W$ is self adjoint under

either of the following conditions : 1) W is the Lie algebra of a compact subgroup G of $SO(q, \mathbb{R})$ or 2) W is defined by a representation of the real, negative definite Clifford algebra $\mathcal{Cl}(p)$ on \mathbb{R}^q . This is equivalent to the condition that Z^2 is a negative multiple of the identity for every nonzero element Z of W . A much more detailed discussion of the structure of \mathfrak{G}_W in special cases may be found in [E7].

In section 7 we relate special properties of the Ricci tensor of a 2-step nilpotent Lie algebra \mathfrak{N} with a suitable inner product $\langle \cdot, \cdot \rangle^*$ to closed orbits of the groups $SL(q, \mathbb{R})$ and $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ in $V = \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$. The action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on V is given by $(g, h)(\sum_{i=1}^N S_i \otimes v_i) = \sum_{i=1}^N g(S_i) \otimes h(v_i)$, where g acts on $\mathfrak{so}(q, \mathbb{R})$ by $g(Z) = gZg^t$. A Zariski open subset V_o of V fibers over $G(p, \mathfrak{so}(q, \mathbb{R}))$, and $GL(p, \mathbb{R})$ acts simply transitively on the fibers.

1. PRELIMINARIES

1.1. Definitions and basic examples.

Definition. A finite dimensional Lie algebra \mathfrak{h} is 2-step nilpotent if \mathfrak{h} is not abelian and $[\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]] = \{0\}$. A Lie group H is 2-step nilpotent if its Lie algebra \mathfrak{h} is 2-step nilpotent.

Clearly, a 2-step nilpotent Lie algebra \mathfrak{h} has a nontrivial center that contains $[\mathfrak{h}, \mathfrak{h}]$. It is reasonable to study 2-step nilpotent Lie groups and Lie algebras. They are as close as possible to being abelian, but the differences from Euclidean space are interesting and challenging. In addition, they arise in other areas of mathematics that are not obviously related, for example as the horospheres of symmetric spaces of strictly negative curvature ([EH, pp. 447-448]). See also (6.1) of [E3] for other examples of 2-step nilpotent Lie algebras.

In this article we will use the letter \mathfrak{N} to denote a nilpotent Lie algebra, typically 2-step, and N will denote the corresponding simply connected nilpotent Lie group with Lie algebra \mathfrak{N} .

Exponential and logarithm functions. It is known that if N is a simply connected nilpotent Lie group with Lie algebra \mathfrak{N} , then $\exp : \mathfrak{N} \rightarrow N$ is a diffeomorphism, where \exp denotes the Lie group exponential map. In this case we let $\log : N \rightarrow \mathfrak{N}$ denote the inverse of the exponential function.

If N is 2-step nilpotent in addition, then the multiplication law in N can be expressed as follows in terms of the exponential map.

$$\exp(X) \cdot \exp(Y) = \exp(X + Y + (1/2)[X, Y]) \text{ for all } X, Y \in \mathfrak{N}.$$

From this formula one quickly obtains additional formulas. Let X, Y be any elements of \mathfrak{N} and write $\phi = \exp(X)$ and $\psi = \exp(Y)$. Then

- a) $\phi \cdot \psi \cdot \phi^{-1} = \exp(Y + [X, Y])$.
- b) $[\phi, \psi] = \phi \cdot \psi \cdot \phi^{-1} \cdot \psi^{-1} = \exp([X, Y])$.
- c) ϕ commutes with $\psi \Leftrightarrow X$ commutes with Y .
- d) $\log(\phi \cdot \psi) = \log \phi + \log \psi + (1/2)[\log \phi, \log \psi]$

1.2. Lattices in nilpotent Lie groups. Let N be a simply connected, nilpotent Lie group, not necessarily with 2-steps, and let \mathfrak{N} denote the Lie algebra of N . A discrete subgroup Γ is called a *lattice* in N if the quotient space $\Gamma \backslash N$ is compact. The quotient space $\Gamma \backslash N$ is called a *compact nilmanifold*, and if N is 2-step nilpotent, then $\Gamma \backslash N$ is called a *compact 2-step nilmanifold*.

A criterion of Mal'cev states that N admits a lattice $\Gamma \Leftrightarrow \mathfrak{N}$ admits a basis \mathfrak{B} with rational structure constants ([Malc]). If N admits a lattice Γ , then $\mathfrak{N}_{\mathbb{Q}} = \mathbb{Q}\text{-span}(\log \Gamma)$ is a Lie algebra over \mathbb{Q} , and any \mathbb{Q} -basis \mathfrak{B} for \mathfrak{N} is also an \mathbb{R} -basis for \mathfrak{N} . Conversely, if \mathfrak{B} is a basis of \mathfrak{N} with rational structure constants and if \mathcal{L} is a vector lattice of \mathfrak{N} contained in $\mathbb{Q}\text{-span}(\mathfrak{B})$, then the subgroup Γ of N generated by $\exp(\mathcal{L})$ is a lattice in N . In this case $\mathbb{Q}\text{-span}(\log \Gamma) = \mathbb{Q}\text{-span}(\mathfrak{B})$.

The correspondence between lattices in \mathbb{N} and rational structures in \mathfrak{N} is almost one-one. In fact, if Γ_1 and Γ_2 are two lattices in \mathbb{N} , then \mathbb{Q} -span($\log \Gamma_1$) = \mathbb{Q} -span($\log \Gamma_2$) \Leftrightarrow Γ_1 and Γ_2 are commensurable; that is, $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2 .

A lattice Γ in \mathbb{N} is called a *full lattice* if $\log \Gamma$ is a vector lattice in \mathfrak{N} ; that is, $\log \Gamma = \mathbb{Q}$ -span(\mathfrak{B}) for some basis \mathfrak{B} of \mathfrak{N} . In this case the basis \mathfrak{B} must have rational structure constants since \mathbb{Q} -span(\mathfrak{B}) = \mathbb{Q} -span($\log \Gamma$) is a Lie algebra over \mathbb{Q} . It is known that for every lattice Γ of \mathbb{N} there exist full lattices Γ_1 and Γ_2 such that $\Gamma_1 \subseteq \Gamma \subseteq \Gamma_2$, Γ_1 has finite index in Γ and Γ has finite index in Γ_2 (Theorem 5.4.2 of [CG]).

1.3. Lie triple systems. We shall see in section 3 that if W is a p -dimensional subspace of $so(q, \mathbb{R})$, the $q \times q$ real, skew symmetric matrices, then $\mathfrak{N} = \mathbb{R}^q \oplus W$ admits the structure of a 2-step nilpotent Lie algebra in a canonical way. However, for a generic choice of W , $\mathfrak{N} = \mathbb{R}^q \oplus W$ has no basis with rational structure constants as we explain in section 5.

A subspace W of $so(q, \mathbb{R})$ is called a *Lie triple system* if $[[W, W], W] \subseteq W$. Clearly a subalgebra W of $so(q, \mathbb{R})$ is a Lie triple system. We define $Z(W) = \{X \in W : [X, Y] = 0 \text{ for all } Y \in W\}$, and we call $Z(W)$ the *center* of W . We say that W has *compact center* if $\exp(Z(W))$ is a compact subset of $SO(q, \mathbb{R})$, where $\exp : so(q, \mathbb{R}) \rightarrow SO(q, \mathbb{R})$ is the matrix exponential map. Note that $\exp(Z(W))$ is always a connected, abelian subgroup of $SO(q, \mathbb{R})$.

Lie triple systems have importance in Riemannian geometry. Let $SO(q, \mathbb{R})$ be equipped with the canonical bi-invariant metric $\langle \cdot, \cdot \rangle$ such that $\langle Z, Z' \rangle = -\text{trace}(ZZ')$ for all Z, Z' in $so(q, \mathbb{R}) = T_e SO(q, \mathbb{R})$. Then a subspace W of $so(q, \mathbb{R})$ is a Lie triple system $\Leftrightarrow \exp(W)$ is a totally geodesic submanifold of $\{SO(q, \mathbb{R}), \langle \cdot, \cdot \rangle\}$. Moreover, every totally geodesic submanifold of $\{SO(q, \mathbb{R}), \langle \cdot, \cdot \rangle\}$ that contains the identity has the form $\exp(W)$ for some Lie triple system W of $so(q, \mathbb{R})$.

Lie triple systems have importance also in the context of simply connected, 2-step nilpotent Lie groups N since they produce examples that admit lattices Γ . The next result is Proposition 4.3c of [E5].

Proposition. *Let W be a Lie triple system in $so(q, \mathbb{R})$ with compact center, and let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be the corresponding 2-step nilpotent Lie algebra. Then \mathfrak{N} admits a basis \mathfrak{B} with rational structure constants.*

Examples of Lie triple systems. The most interesting examples of Lie triple systems W in $so(q, \mathbb{R})$ arise from the representations of algebras, both Clifford algebras and compact semisimple Lie algebras. The latter case corresponds precisely to the set of subalgebras W of $so(q, \mathbb{R})$ that have trivial center. The representations of Clifford algebras lead to the examples $\mathfrak{N} = \mathbb{R}^q \oplus W$ of *Heisenberg type*, and the representations of compact semisimple Lie algebras lead to examples $\mathfrak{N} = \mathbb{R}^q \oplus W$ that are *naturally reductive*. We omit further details of these two main examples. For results about the examples of Heisenberg type see [BTV], [CD], [CDKR], [DR], [E1], [E5], [K1], [K2], [La5], [Pe] and [R]. For results about the examples arising from representations of compact, semisimple Lie algebras see [EH], [Go], [La4] and [La6]. We shall return to these examples later in the discussion of the Ricci tensor.

A brief classification of Lie triple systems. The classification of Lie triple systems in $so(q, \mathbb{R})$ that have trivial center reduces to the problem of classifying involutive automorphisms of compact semisimple Lie algebras. We explain this with three examples and a short result that ties them together.

Example 1 W is a subalgebra of $so(q, \mathbb{R})$.

Example 2 Let \mathfrak{G} be a subalgebra of $\mathfrak{so}(q, \mathbb{R})$ with trivial center (i.e. a semisimple subalgebra). Let $\sigma : \mathfrak{G} \rightarrow \mathfrak{G}$ be an automorphism such that $\sigma^2 = \text{Id}$. If $W \subseteq \mathfrak{G}$ is the -1 eigenspace of σ , then W is a Lie triple system in $\mathfrak{so}(q, \mathbb{R})$.

For a classification of these involutive automorphisms see, for example, [Hel, pp. 451-455].

Example 3 Let \mathfrak{G} be a semisimple subalgebra of $\mathfrak{so}(q, \mathbb{R})$, and suppose that $\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$, where \mathfrak{G}_1 and \mathfrak{G}_2 are semisimple ideals of \mathfrak{G} . Let W_2 be the -1 eigenspace of an involutive automorphism $\sigma : \mathfrak{G}_2 \rightarrow \mathfrak{G}_2$. Then $W = \mathfrak{G}_1 \oplus W_2$ is a Lie triple system in $\mathfrak{so}(q, \mathbb{R})$.

If W is a Lie triple system with trivial center in $\mathfrak{so}(q, \mathbb{R})$, then $\mathfrak{h} = W + [W, W]$ is a semisimple subalgebra of $\mathfrak{so}(q, \mathbb{R})$. See Lemma 1 in the proof of Proposition 4.3c of [E5] for details. In fact, the proof of Proposition 4.3c of [E5] leads to the following result, whose proof we omit.

Proposition. *Let W be a Lie triple system of $\mathfrak{so}(q, \mathbb{R})$ that has trivial center. Then W arises as in example 1, 2 or 3.*

1.4. The maps $j(Z)$. Let \mathfrak{N} be a 2-step nilpotent Lie algebra with a positive definite inner product $\langle \cdot, \cdot \rangle$. Let N denote the simply connected, 2-step nilpotent Lie group with Lie algebra \mathfrak{N} , and let $\langle \cdot, \cdot \rangle$ also denote the left invariant Riemannian metric on N that is determined by the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{N} = T_e N$.

Following Kaplan [K1, 2] we decompose \mathfrak{N} into an orthogonal direct sum $\mathfrak{N} = V \oplus \mathfrak{Z}$ where \mathfrak{Z} denotes the center of \mathfrak{N} and $V = \mathfrak{Z}^\perp$. For each $Z \in \mathfrak{Z}$ we define a skew symmetric linear transformation $j(Z) : V \rightarrow V$ by $j(Z)X = (\text{ad } X)^*(Z)$, where $(\text{ad } X)^*$ denotes the transpose of $\text{ad } X : \mathfrak{N} \rightarrow \mathfrak{Z}$ determined by $\langle \cdot, \cdot \rangle$.

Equivalently, $j(Z) : V \rightarrow V$ is the unique linear map such that $\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle$ for all $X, Y \in V$ and $Z \in \mathfrak{Z}$. It is evident that $j : \mathfrak{Z} \rightarrow \mathfrak{so}(V, \langle \cdot, \cdot \rangle)$ is a linear map.

If a skew symmetric transformation Z leaves invariant a subspace U of V , then Z leaves invariant the orthogonal complement U^\perp in V . Hence V can always be written as an orthogonal direct sum of subspaces invariant under $j(\mathfrak{Z}) = \{j(Z) : Z \in \mathfrak{Z}\}$. We say that $j(\mathfrak{Z})$ acts *irreducibly* on V if $j(\mathfrak{Z})$ leaves invariant no proper subspace of V .

In principle one can reduce the study of metric 2-step nilpotent Lie algebras $\mathfrak{N} = V \oplus \mathfrak{Z}$ to the case that $j(\mathfrak{Z})$ acts irreducibly on V . We shall not pursue this idea although we find it frequently convenient in this article to assume that $j(\mathfrak{Z})$ acts irreducibly.

Much of the geometry of N can be described by the maps $j(\mathfrak{Z})$. See [K1, 2] and [E1] for further details. We present one example that is particularly relevant for this article.

1.5. Ricci tensor. If M is any Riemannian manifold, then the Ricci tensor is a symmetric, bilinear form on each tangent space of M defined by the formula $\text{Ric}(v, w) = \text{trace}(R_{vw})$, where $(R_{vw})(z) = R(z, v)w$ and R denotes the curvature tensor of M .

Let H be a connected Lie group with a left invariant metric $\langle \cdot, \cdot \rangle$ arising from an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra \mathfrak{h} . Then the Ricci tensor Ric , like the curvature tensor R , may be defined by its values on left invariant vector fields of H , or equivalently, by its values in $\mathfrak{h} = T_e H$. Hence we may regard the Ricci tensor as a symmetric, bilinear map $\mathbf{Ric} : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{R}$ given by the formula above.

If the Lie algebra \mathfrak{h} is 2-step nilpotent, then we can be more specific. See Proposition 2.5 of [E1] for a proof of the next result.

Proposition. *Let N be a simply connected, 2-step nilpotent Lie group with a left invariant metric $\langle \cdot, \cdot \rangle$, and let \mathfrak{N} denote the Lie algebra of N . Write $\mathfrak{N} = V \oplus \mathfrak{Z}$, where \mathfrak{Z} denotes the center of \mathfrak{N} and $V = \mathfrak{Z}^\perp$ denotes the orthogonal complement of \mathfrak{Z} . Then*

- 1) $\mathbf{Ric}(X, Z) = 0$ for all $X \in V$ and all $Z \in \mathfrak{Z}$.
- 2) For $X, Y \in V$, $\mathbf{Ric}(X, Y) = \langle T(X), Y \rangle$, where $T : V \rightarrow V$ is the

symmetric linear transformation given by $T = (1/2) \sum_{i=1}^p j(Z_i)^2$, where $\{Z_1, \dots, Z_p\}$ is any orthonormal basis of \mathfrak{Z} . In particular T is negative definite on V .
 3) For $Z, Z^* \in \mathfrak{Z}$, $\mathbf{Ric}(Z, Z^*) = - (1/4)$ trace $j(Z) \circ j(Z^*)$. In particular $\mathbf{Ric}(Z, Z) \geq 0$ for all $Z \in \mathfrak{Z}$ with equality $\Leftrightarrow j(Z) = 0$.

Remark. The statement above says that the Ricci tensor is negative definite on V , and positive semidefinite on \mathfrak{Z} . It follows from Proposition 2.7 of [E1] that the Ricci tensor is positive definite on \mathfrak{Z} if N has no Euclidean de Rham factor. This means that the Ricci tensor of a left invariant metric on N can never be positive semidefinite or negative semidefinite. Comparison theorem techniques (cf. [CE]), which have played a central role in studying Riemannian manifolds with sectional or Ricci curvatures of a fixed sign, cannot be used to study the geometry of 2-step nilpotent Lie groups N with a left invariant metric. As a consequence the geometry of such groups is an interesting mixture of phenomena that occur in spaces of positive, negative and zero sectional curvature.

Optimal Ricci tensors. Let $\{\mathfrak{N}, \langle, \rangle\}$ be a 2-step nilpotent, metric Lie algebra. Write $\mathfrak{N} = V \oplus \mathfrak{Z}$ where \mathfrak{Z} denotes the center of \mathfrak{N} and $V = \mathfrak{Z}^\perp$ denotes the orthogonal complement of \mathfrak{Z} . The Ricci tensor of $\{\mathfrak{N}, \langle, \rangle\}$ is said to be *optimal* if there exist positive numbers λ and μ such that

- 1) $\mathbf{Ric}(X, Y) = -\lambda^2 \langle X, Y \rangle$ for all X, Y in V .
- 2) $\mathbf{Ric}(Z, Z^*) = \mu^2 \langle Z, Z^* \rangle$ for all Z, Z^* in \mathfrak{Z} .

In view of the preceding remark and the proposition before it the use of the word optimal seems appropriate. We shall see later in section 7 that 2-step nilpotent Lie algebras that admit metrics \langle, \rangle with an optimal Ricci tensor can be characterized by the closed orbits of $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ in the vector space $V = so(q, \mathbb{R})^p = so(q, \mathbb{R}) \times \dots \times so(q, \mathbb{R})$ (p times).

Geodesic flow invariant Ricci tensors. Let $\{\mathfrak{N}, \langle, \rangle\}$ be a 2-step nilpotent, metric Lie algebra, and let $\{N, \langle, \rangle\}$ denote the corresponding simply connected, 2-step nilpotent Lie group with Lie algebra \mathfrak{N} and left invariant metric \langle, \rangle determined by the inner product \langle, \rangle on \mathfrak{N} .

Let $\{g^t\}$ denote the geodesic flow of $\{N, \langle, \rangle\}$ in TN . We say that the Ricci tensor of $\{N, \langle, \rangle\}$ is *geodesic flow invariant* if $\mathbf{Ric}(g^t v, g^t v) = \mathbf{Ric}(v, v)$ for all vectors v in TN and all real numbers t .

We shall see later in section 7 that 2-step nilpotent Lie algebras that admit metrics \langle, \rangle with a geodesic flow invariant Ricci tensor can be characterized by the closed orbits of $SL(q, \mathbb{R})$ in the vector space $V = so(q, \mathbb{R})^p$.

2. POISSON STRUCTURE AND THE GEODESIC FLOW

We summarize some basic facts and examples about Poisson manifolds. For further information see the survey article [E3] or one of the sources listed in the bibliography of that article (e.g. [MR], [OI]).

2.1. Definition and properties of the symplectic 2-form.

The symplectic 2-form on T^*M . Let M be a C^∞ manifold with tangent bundle TM and cotangent bundle T^*M . The cotangent bundle admits a canonical 1-form θ defined by $\theta(\xi) = \omega(d\pi(\xi))$, where ξ is an element of $T_\omega(T^*M)$ and $\pi: T^*M \rightarrow M$ is the projection that assigns to an element ω of $(T_m M)^*$ the base point m .

The 2-form $\Omega = -d\theta$ is a symplectic form on T^*M ; that is, Ω is nondegenerate at every point of T^*M and $\Omega \wedge \dots \wedge \Omega$ (n times) is a nonzero $2n$ -form at every point of T^*M .

The symplectic 2-form on TM. Now let M be a Riemannian manifold, and let $\langle \cdot, \cdot \rangle_m$ denote the inner product on each tangent space $T_m M$. The bilinear form $\langle \cdot, \cdot \rangle$ defines an isomorphism f_m between $T_m M$ and $T_m M^*$ for each point m of M by $f_m(v)(w) = \langle v, w \rangle$ for all vectors v, w in $T_m M$. The resulting diffeomorphism $f = f_{\langle \cdot, \cdot \rangle} : TM \rightarrow T^*M$ allows one to pull back the 1-form θ and the 2-form $\Omega = -d\theta$ on T^*M . These pullbacks will also be denoted θ and Ω . It is routine to show

$$(1) \theta(\xi) = \langle v, d\pi(\xi) \rangle \text{ for every } \xi \in T_v(TM)$$

where $\pi : TM \rightarrow M$ denotes the projection map that sends a vector $v \in T_m M$ to the base point m . Since d commutes with pullbacks we also have

$$(2) \Omega = -d\theta$$

2.2. Poisson structures on a manifold.

Definition. A C^∞ manifold P is called a *Poisson manifold* if there is a structure $\{, \} : C^\infty(P) \times C^\infty(P) \rightarrow C^\infty(P)$ that satisfies the following properties :

- 1) (Skew symmetry) $\{f, g\} = -\{g, f\}$ for all $f, g \in C^\infty(P)$
- 2) (Bilinearity) $\{, \}$ is \mathbb{R} -bilinear
 - a) $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$ for $a, b \in \mathbb{R}$ and $f, g, h \in C^\infty(P)$
 - b) $\{f, ag + bh\} = a\{f, g\} + b\{f, h\}$ for $a, b \in \mathbb{R}$ and $f, g, h \in C^\infty(P)$
- 3) (Leibniz) $\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$ for all $f, g, h \in C^\infty(P)$
- 4) (Jacobi Identity) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ for all $f, g, h \in C^\infty(P)$

Hamiltonian vector fields. If we are given a Poisson structure $\{, \}$ on $C^\infty(P)$, then for each $f \in C^\infty(P)$ we may define a function $X_f : C^\infty(P) \rightarrow C^\infty(P)$ by $X_f(g) = \{g, f\}$. The Leibniz property then becomes the statement $X_f(g \cdot h) = (X_f g) \cdot h + g \cdot (X_f h)$ for all $f, g, h \in C^\infty(P)$. In particular, each map X_f is a derivation on $C^\infty(P)$ and consequently X_f defines an element of $\mathfrak{X}(P)$, the space of C^∞ vector fields on P . The vector field X_f is called the *Hamiltonian vector field* on P determined by f .

Example (Geodesic vector field). Let M be a Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle$. Let $\Omega = -d\theta$ be the natural symplectic 2-form defined above. Let $\{, \}$ be the Poisson structure on the tangent bundle TM defined by Ω as in Example 1 below. Let $E : TM \rightarrow \mathbb{R}$ be the energy function defined by $E(v) = (1/2) \langle v, v \rangle$. Then the Hamiltonian vector field X_E is the vector field on M whose flow transformations $\{g^t\}$ define the geodesic flow on TM ; that is, if v is any vector in TM and if $\gamma_v(t)$ denotes the geodesic of M with initial velocity v , then $\gamma_v'(t) = g^t(v)$ for all $t \in \mathbb{R}$.

Examples of Poisson manifolds.

Example 1. Let P be a C^∞ manifold with a symplectic 2-form Ω . For every point x in P and every $\omega \in T_x^*P$ there is a unique vector $\xi \in T_x P$ such that $\omega(\xi^*) = \Omega(\xi, \xi^*)$ for all $\xi^* \in T_x^*P$; this follows from the fact that Ω is nondegenerate at every point of P . In particular, for every $f \in C^\infty(P)$ there exists a unique C^∞ vector field X_f such that $\Omega(X_f, \cdot) = df$. Now define $\{f, g\} = \Omega(X_f, X_g)$. It is not difficult to check that $\{, \}$ satisfies the Poisson axioms on P and X_f is the Hamiltonian vector field associated to f by the Poisson structure $\{, \}$.

Example 2. Let \mathfrak{h} be a finite dimensional real Lie algebra. The Lie algebra \mathfrak{h} may be regarded as the subspace of linear functions in $C^\infty(\mathfrak{h}^*)$ under the natural isomorphism between \mathfrak{h} and $(\mathfrak{h}^*)^*$: given $A \in \mathfrak{h}$ and $\omega \in \mathfrak{h}^*$ define $A(\omega) = \omega(A)$. Define $\{, \}$ on $C^\infty(\mathfrak{h}^*)$ by first requiring that $\{A, B\} = [A, B]$ for all A, B in \mathfrak{h} . Then there is a unique extension of $\{, \}$ from \mathfrak{h} to all of $C^\infty(\mathfrak{h}^*)$. Note that \mathfrak{h} is a Lie subalgebra of $C^\infty(\mathfrak{h}^*)$.

Example 3. Let \mathfrak{h} be a finite dimensional real Lie algebra, and let \langle , \rangle be a nondegenerate, symmetric bilinear form on \mathfrak{h} . Let $\# : \mathfrak{h} \rightarrow \mathfrak{h}^*$ be the isomorphism defined by $A^\#(B) = \langle A, B \rangle$ for all $A, B \in \mathfrak{h}$. Define $\{, \}^\#$ on $\mathfrak{h}^* \subseteq C^\infty(\mathfrak{h})$ by $\{A^\#, B^\#\}^\# = [A, B]^\#$ for all $A, B \in \mathfrak{h}$. Then $\{, \}^\#$ has a unique extension to a Poisson structure on \mathfrak{h} . As in the previous example, we note that \mathfrak{h}^* is a Lie subalgebra of $C^\infty(\mathfrak{h})$.

Poisson maps and anti-Poisson maps. Let P_1 and P_2 be Poisson manifolds with Poisson structures $\{, \}_1$ and $\{, \}_2$. A C^∞ map $\varphi : P_1 \rightarrow P_2$ is called a *Poisson map* if $\{f \circ \varphi, g \circ \varphi\}_1 = \{f, g\}_2$ for all $f, g \in C^\infty(P_2)$. If $P_1 = P_2$ and φ is a diffeomorphism, then φ is called a *Poisson automorphism*.

We call a C^∞ map $\varphi : P_1 \rightarrow P_2$ an *anti Poisson map* if $\{f \circ \varphi, g \circ \varphi\}_1 = -\{f, g\}_2$ for all $f, g \in C^\infty(P_2)$.

2.3. Poisson structures associated to a Lie group.

The tangent bundle TH and the Gauss map $G : TH \rightarrow \mathfrak{h}$. Let H be a Lie group with Lie algebra \mathfrak{h} and tangent bundle TH . Define the left translation L_h on H by $L_h(h^*) = hh^*$ for $h, h^* \in H$. This defines a natural left action of H on itself. Identify H with T_eH , the tangent space to H at the identity, and identify TH with $H \times \mathfrak{h}$ under the diffeomorphism $(h, X) \rightarrow dL_h(X)$.

Define $G : TH = H \times \mathfrak{h} \rightarrow \mathfrak{h}$ by $G(h, X) = X$, projection onto the second factor. Geometrically, this amounts to the left translation of a vector $\xi \in TH$ back to the identity. In the case that H is the abelian Lie group \mathbb{R}^3 with vector addition as the group operation, then the Gauss map defined here is precisely the classical Gauss map used to study the geometry of surfaces in \mathbb{R}^3 .

Now let H be a Lie group with Lie algebra \mathfrak{h} and let \langle , \rangle be a positive definite inner product on \mathfrak{h} . From \langle , \rangle we obtain Poisson structures on \mathfrak{h} (Example 3) and TH (Example 1). The Gauss map $G : TH \rightarrow \mathfrak{h}$ is an anti Poisson map for every choice of inner product \langle , \rangle on \mathfrak{h} . See Proposition A of (4.4f) of [E3] for a proof.

Remark. The Gauss map $G : TH \rightarrow \mathfrak{h}$ does not depend on a choice of \langle , \rangle , and it acts as an anti Poisson map for every choice of \langle , \rangle . The Gauss map is an important tool for the study of the geodesic flow on the unit tangent bundle SH of a Lie group H with a left invariant Riemannian metric arising from a choice of positive definite inner product \langle , \rangle on \mathfrak{h} .

Left invariant functions on TH. A function $f : TH \rightarrow \mathbb{R}$ will be called *left invariant* if f is constant on all H -orbits in $TH = H \times \mathfrak{h}$; that is, $f(h, X) = f(e, X)$ for all $h \in H$ and all $X \in \mathfrak{h}$.

The left invariant functions on TH may be identified with the functions on \mathfrak{h} . If $f : TH \rightarrow \mathbb{R}$ is a left invariant function, then we may define $\mathbf{f} : \mathfrak{h} \rightarrow \mathbb{R}$ by $\mathbf{f}(X) = f(e, X)$, where e denotes the identity in H . Conversely, if we are given a function $\mathbf{f} : \mathfrak{h} \rightarrow \mathbb{R}$, then \mathbf{f} arises from a left invariant function $f : TH \rightarrow \mathbb{R}$ given by $f(h, X) = \mathbf{f}(X)$ for all $(h, X) \in H \times \mathfrak{h}$. Note that $f = \mathbf{f} \circ G$.

Example. Let $E : TH \rightarrow \mathbb{R}$ be the energy function given by $E(v) = (1/2) \langle v, v \rangle$, where \langle , \rangle is the left invariant metric arising from an inner product \langle , \rangle on \mathfrak{h} . The function E is left invariant, and the corresponding function \mathbf{E} is just the restriction of E to $\mathfrak{h} = T_eH$.

Hamiltonian vector fields of left invariant functions.

Proposition. Let H be a connected Lie group with Lie algebra \mathfrak{h} and a left invariant metric \langle , \rangle . Let $f : TH \rightarrow \mathbb{R}$ be a left invariant function with companion function $\mathbf{f} : \mathfrak{h} \rightarrow \mathbb{R}$. Let X_f and $X_{\mathbf{f}}$ denote the Hamiltonian vector fields defined by the Poisson structures on TH and \mathfrak{h} . Let $G : TH \rightarrow \mathfrak{h}$ be the Gauss map. Then

$$G_*(X_f(\xi)) = -X_{\mathbf{f}}(G(\xi)) \quad \text{for all } \xi \in TH.$$

The minus sign that occurs is due to the fact that $G : TH \rightarrow \mathfrak{H}$ is an anti-Poisson map. This result will be useful in looking for first integrals of the geodesic flow $\{g^t\}$ on TH. Recall that the corresponding geodesic vector field is X_E , where E is the energy function.

2.4. First integrals for the geodesic flow. Let $\{\varphi^t\}$ denote a flow on a manifold X . For simplicity we shall assume that both X and $\{\varphi^t\}$ are C^∞ . A function $f : X \rightarrow \mathbb{R}$ is said to be a *first integral* for the flow $\{\varphi^t\}$ if $f(\varphi^t(x)) = f(x)$ for all $t \in \mathbb{R}$. Finding nonconstant first integrals for a flow $\{\varphi^t\}$ is an important tool for understanding the orbit structure of the flow. Clearly if f is a first integral for a flow $\{\varphi^t\}$ and $f(x) \neq f(y)$ for points x, y in X , then x and y lie on different orbits of the flow. The goal, which is usually not realized, is to find a finite set of first integrals $\{f_1, \dots, f_N\}$ such that points x, y of X lie in the same orbit of $\{\varphi^t\} \Leftrightarrow f_i(x) = f_i(y)$ for $1 \leq i \leq N$.

Now let H be a connected Lie group with Lie algebra \mathfrak{H} . Let $\langle \cdot, \cdot \rangle$ denote an inner product on \mathfrak{H} and also the corresponding left invariant metric on H . Let $\{, \}$ denote the Poisson structure on both TH and \mathfrak{H} defined by $\langle \cdot, \cdot \rangle$. Note that if $f : TH \rightarrow \mathbb{R}$ is a C^∞ function, then the first integrals for the Hamiltonian vector field X_f are precisely the functions $g : TH \rightarrow \mathbb{R}$ such that $\{g, f\} = 0$.

Geodesic flow in TH and \mathfrak{H} . We regard the geodesic vector field on TH as the Hamiltonian vector field X_E , where $E : TH \rightarrow \mathbb{R}$ is the energy function. We define the geodesic vector field in \mathfrak{H} to be the Hamiltonian vector field $X_{\mathbf{E}}$, where $\mathbf{E} : \mathfrak{H} \rightarrow \mathbb{R}$ is the restriction of E to $\mathfrak{H} = T_e H$. The flow transformations of X_E and $X_{\mathbf{E}}$ will be called the *geodesic flow* in TH and \mathfrak{H} respectively.

In the context of Hamiltonian vector fields there are two important types of first integrals for the vector field X_E on TH. Although these examples provide a lot of information about first integrals on TH they seem to fall short of determining $\{, \}$, except in very special cases, when the geodesic flow on TH is completely integrable.

1) Universal first integrals. Let θ denote the canonical 1-form on TH pulled back from the canonical 1-form on T^*H by the Riemannian metric $\langle \cdot, \cdot \rangle$. Let $\lambda : H \rightarrow \text{Diff}(TH)$ be the homomorphism such that $\lambda_h(v) = dL_h(v)$, where $L_h : H \rightarrow H$ denotes left translation by h . If $\mathfrak{X}(TH)$ denotes the Lie algebra of C^∞ vector fields on TH, then λ defines a Lie algebra anti homomorphism $\lambda : \mathfrak{H} \rightarrow \mathfrak{X}(TH)$ such that the flow transformations of $\lambda(A)$ in TH are $\{\lambda_{\varphi^t A}\}$ for all $A \in \mathfrak{H}$.

The next result is well known. See also sections 3.11 and 5.3 of [E3].

Proposition 2.4.1. *Let $f : TH \rightarrow \mathbb{R}$ be a left invariant C^∞ function. For any $A \in \mathfrak{H}$ the function $g_A = \theta(\lambda(A)) : TH \rightarrow \mathbb{R}$ is a first integral for the Hamiltonian vector field X_f .*

In particular each function g_A is a first integral for the geodesic vector field X_E . The functions g_A are not left invariant in general unless A lies in the center of \mathfrak{H} .

2) Left invariant first integrals.

Proposition 2.4.2. *Let $f : TH = H \times \mathfrak{H} \rightarrow \mathbb{R}$ be a left invariant function, and let $\mathbf{f} : \mathfrak{H} \rightarrow \mathbb{R}$ denote its restriction to \mathfrak{H} . Then f is a first integral for the geodesic flow in TH $\Leftrightarrow \mathbf{f}$ is a first integral for the geodesic flow in $\mathfrak{H} \Leftrightarrow \{\mathbf{f}, \mathbf{E}\} = 0$.*

Proof. Since the Gauss map $G : TH \rightarrow \mathfrak{H}$ is an anti Poisson map it follows that $\{f, E\} = \{\mathbf{f} \circ G, \mathbf{E} \circ G\} = -\{\mathbf{f}, \mathbf{E}\} \circ G$, where $\mathbf{E} : \mathfrak{H} \rightarrow \mathbb{R}$ denotes the restriction of E to \mathfrak{H} . The function f is a first integral for the geodesic flow in TH $\Leftrightarrow \{f, E\} = 0 \Leftrightarrow \{\mathbf{f}, \mathbf{E}\} = 0 \Leftrightarrow \mathbf{f}$ is a first integral for the geodesic flow in \mathfrak{H} . \square

The result above says that every first integral for the geodesic flow in \mathfrak{H} produces a left invariant first integral for the geodesic flow in TH. See (5.3b) of [E3] for some discussion of first integrals for the geodesic flow in \mathfrak{H} when \mathfrak{H} is an arbitrary Lie algebra with inner product $\langle \cdot, \cdot \rangle$. In the most

extreme case $\{\mathbf{f}, \mathbf{E}\} = 0$ for every C^∞ function $\mathbf{f}: \mathfrak{N} \rightarrow \mathbb{R}$ if \mathfrak{N} is a Lie algebra with negative definite Killing form \mathbf{B} and $\langle \cdot, \cdot \rangle = -\mathbf{B}$.

2.5. First integrals for the geodesic flow in the 2-step nilpotent case. We describe some first integrals for the geodesic flow on a simply connected, 2-step nilpotent Lie group N with a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$. These results are discussed in more detail in (6.8) of [E3]. In this section we use the decomposition $\mathfrak{N} = V \oplus \mathfrak{Z}$ described above in (1.4), where \mathfrak{Z} denotes the center of \mathfrak{N} and V denotes the orthogonal complement of \mathfrak{Z} in \mathfrak{N} . We recall also that each element Z of \mathfrak{Z} defines a skew symmetric map $j(Z): V \rightarrow V$ given by $\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle$ for all X, Y in V .

Nonsingular and almost nonsingular 2-step nilpotent Lie algebras. Let \mathfrak{N} be a 2-step nilpotent Lie algebra. We say that \mathfrak{N} is *nonsingular* if $\text{ad } \xi: \mathfrak{N} \rightarrow \mathfrak{Z}$ is surjective for all $\xi \in \mathfrak{N} - \mathfrak{Z}$. We say that \mathfrak{N} is *almost nonsingular* if there exists $\omega \in \mathfrak{N}^*$ such that if $\omega \circ \text{ad } X$ is identically zero on \mathfrak{N} , then $X \in \mathfrak{Z}$, the center of \mathfrak{N} .

Nonsingular and almost nonsingular Lie algebras \mathfrak{N} are the most well behaved of all 2-step nilpotent Lie algebras. It is easy to show that every nonsingular 2-step nilpotent Lie algebra is almost nonsingular. See (6.8) of [E3] for further discussion.

Description of the geodesic flow. We first describe the geodesic flow $\{g^t\}$ of the geodesic vector field X_E in TN by using the identification of TN with $N \times \mathfrak{N}$ that we described above in (2.3). The next result is a restatement of Proposition 3.2 of [E1].

Proposition 2.5.1. *Let $\xi = (n, C) \in TN$. Let $\gamma_\xi(t)$ denote the geodesic of N with initial velocity ξ , and write $C = X + Z$, where $X \in V$ and $Z \in \mathfrak{Z}$. Then $g^t(n, C) = (\gamma_\xi(t), e^{tj(Z)}X + Z)$ for all $t \in \mathbb{R}$.*

As a consequence of the result above and the proposition in (2.3) we obtain a description of the geodesic flow maps $\{g^t\}$ of the geodesic vector field X_E in \mathfrak{N} .

Corollary 2.5.2. *Let $\{g^t\}$ be the flow maps of the geodesic vector field X_E in \mathfrak{N} . Then $g^t(X + Z) = e^{-tj(Z)}X + Z$ for all $X \in V$, $Z \in \mathfrak{Z}$ and $t \in \mathbb{R}$.*

We use the result above to describe the zero locus of the geodesic vector field X_E in \mathfrak{N} .

Corollary 2.5.3. *Let X_E be the geodesic vector field in \mathfrak{N} . Then $X_E(\xi) = 0$ if $\xi \in V \cup \mathfrak{Z}$. If \mathfrak{N} is nonsingular, then $V \cup \mathfrak{Z} = \{\xi \in \mathfrak{N} : X_E(\xi) = 0\}$.*

The canonical \mathfrak{Z} -valued first integral. Identifying TN with $N \times \mathfrak{N}$ as above we define a map $\varphi: TN \rightarrow \mathfrak{Z}$ by

$$\varphi(n, X + Z) = Z \text{ for } n \in N, X \in V \text{ and } Z \in \mathfrak{Z}.$$

It is clear from the previous proposition that φ is a \mathfrak{Z} -valued first integral for the geodesic flow $\{g^t\}$ in TN .

The next result relates the first integral $\varphi: TN \rightarrow \mathfrak{Z}$ to the universal first integrals defined in Proposition 2.4.1. For details of the proof see (6.8b) of [E3].

Proposition 2.5.4. *Let $\varphi: TN \rightarrow \mathfrak{Z}$ be the canonical \mathfrak{Z} -valued first integral for the geodesic flow $\{g^t\}$ in TN . Then $\langle \varphi(\xi), A \rangle = \theta(\lambda(A))(\xi)$ for all $A \in \mathfrak{Z}$ and all $\xi \in TN$.*

Polynomial first integrals for the geodesic vector field in \mathfrak{N} . As a simple first step we look for polynomial functions $\mathbf{f}: \mathfrak{N} \rightarrow \mathbb{R}$ that are first integrals for the geodesic flow $\{g^t\}$ of the geodesic vector X_E in \mathfrak{N} . Equivalently, we look for polynomial functions $\mathbf{f}: \mathfrak{N} \rightarrow \mathbb{R}$ such that $\{\mathbf{f}, \mathbf{E}\} = 0$. It is not difficult to show that one may reduce to the case that \mathbf{f} is homogeneous of a given degree. See (5.3b) of [E3] for further details.

Linear first integrals.

Proposition 2.5.5. *Let \mathfrak{N} be a 2-step nilpotent Lie algebra with an inner product $\langle \cdot, \cdot \rangle$, and let $\# : \mathfrak{N} \rightarrow \mathfrak{N}^*$ be the canonical isomorphism given by $A^\#(B) = \langle A, B \rangle$ for $A, B \in \mathfrak{N}$. For an element A of \mathfrak{N} let $\mathbf{f} = A^\# : \mathfrak{N} \rightarrow \mathbb{R}$. Then*

1) $\{\mathbf{f}, \mathbf{E}\} = 0 \Leftrightarrow A \in \mathfrak{Z}$, the center of \mathfrak{N} .

2) If $\{\mathbf{f}, \mathbf{E}\} = 0$, then \mathbf{f} is a Casimir function; that is, $\{\mathbf{f}, \mathbf{g}\} = 0$ for all C^∞ functions $\mathbf{g} : \mathfrak{N} \rightarrow \mathbb{R}$, or equivalently the Hamiltonian vector field $X_{\mathbf{f}}$ is identically zero in \mathfrak{N} .

Quadratic first integrals. We consider next the homogeneous polynomials of degree 2. Any such polynomial function $\mathbf{f} : \mathfrak{N} \rightarrow \mathbb{R}$ can be expressed as $\mathbf{f}(\xi) = \langle S(\xi), \xi \rangle$, where $S : \mathfrak{N} \rightarrow \mathfrak{N}$ is a linear map that is symmetric relative to $\langle \cdot, \cdot \rangle$. By the discussion in (5.3b) of [E3] the polynomial is a first integral for the geodesic flow \mathbf{g}^t in $\mathfrak{N} \Leftrightarrow$

$$(*) \quad \langle S(A), \nabla_A A \rangle = 0 \text{ for all } A \in \mathfrak{N}.$$

Let $A \in \mathfrak{N}$ and write $A = X + Z$, where $X \in V$ and $Z \in \mathfrak{Z}$. From (2.2) of [E1] it follows that $\nabla_A A = -j(Z)X$.

Example The Ricci tensor. The Ricci tensor $\mathbf{Ric} : \mathfrak{N} \rightarrow \mathbb{R}$ is a quadratic polynomial function defined by $\mathbf{Ric}(\xi) = \text{Ric}(\xi, \xi) = \langle T(\xi), \xi \rangle$, where $T : \mathfrak{N} \rightarrow \mathfrak{N}$ is a symmetric linear transformation. By the discussion in (1.4) the map T leaves invariant both V and \mathfrak{Z} .

The next result leads to a necessary and sufficient condition for $\mathbf{Ric} : \mathfrak{N} \rightarrow \mathbb{R}$ to be a first integral for the geodesic flow in \mathfrak{N} .

Proposition 2.5.6. *Let \mathfrak{N} be a 2-step nilpotent Lie algebra. Let $S : \mathfrak{N} \rightarrow \mathfrak{N}$ be a symmetric linear transformation and define $\mathbf{f} : \mathfrak{N} \rightarrow \mathbb{R}$ by $\mathbf{f}(x) = \langle S(\xi), \xi \rangle$ for all $\xi \in \mathfrak{N}$. Then \mathbf{f} is a first integral for the geodesic flow $\{\mathbf{g}^t\}$ in \mathfrak{N} if both of the following conditions hold*

1) $S(V) \subseteq V$ and $S(\mathfrak{Z}) \subseteq \mathfrak{Z}$.

2) S commutes with $j(Z)$ on V for all $Z \in \mathfrak{Z}$.

Conversely, if \mathfrak{N} is almost nonsingular and \mathbf{f} is a first integral for the geodesic flow \mathbf{g}^t in \mathfrak{N} , then S satisfies conditions 1) and 2).

Remark. This result is stated slightly differently in (6.8) of [E3], but the proof of that result proves the statement above as well.

First integrals and the Ricci tensor. As an application of the result above we obtain

Proposition 2.5.7. *Let $\mathfrak{N} = V \oplus \mathfrak{Z}$ and let $T = (1/2) \sum_{i=1}^p j(Z_i)^2$, where $\{Z_1, \dots, Z_p\}$ is an orthonormal basis for \mathfrak{Z} . The Ricci tensors $\mathbf{Ric} : \mathfrak{N} \rightarrow \mathbb{R}$ and $\text{Ric} : TN \rightarrow \mathbb{R}$ are first integrals for the geodesic flow in \mathfrak{N} and TN respectively $\Leftrightarrow T$ commutes with $j(Z)$ on V for all $Z \in \mathfrak{Z}$.*

Proof. By Proposition 2.4.2 $\mathbf{Ric} : \mathfrak{N} \rightarrow \mathbb{R}$ is a first integral for the geodesic flow in $\mathfrak{N} \Leftrightarrow \text{Ric} : TN \rightarrow \mathbb{R}$ is a first integral for the geodesic flow in TN . By the discussion in (1.4) $\mathbf{Ric}(\xi, \eta) = \langle T(\xi), \eta \rangle$ for all $\xi, \eta \in V$, where T is defined above.

If T commutes with $j(Z)$ on V for all $Z \in \mathfrak{Z}$, then $\mathbf{Ric} : \mathfrak{N} \rightarrow \mathbb{R}$ is a first integral for the geodesic flow in \mathfrak{N} by Proposition 2.5.6 and the remarks preceding it.

Suppose now that $\mathbf{Ric} : \mathfrak{N} \rightarrow \mathbb{R}$ is a first integral for the geodesic flow in \mathfrak{N} . To prove that T commutes with $j(Z)$ on V for all $Z \in \mathfrak{Z}$ we will need the following

Lemma. *Let N be a simply connected, 2-step nilpotent Lie group with a left invariant metric $\langle \cdot, \cdot \rangle$. Let $c(t)$ be a geodesic of N with $c(0) = e$, the identity of N , and $c'(0) = X + Z$, where $X \in V$ and $Z \in \mathfrak{Z}$. Let $J = j(Z) : V \rightarrow V$. Then*

$$\mathbf{Ric}(c'(t), c'(t)) = \langle Te^{tJ}X, e^{tJ}X \rangle - (1/4) \text{trace } J^2.$$

We suppose for the moment that the lemma has been proved, and we complete the proof of the proposition. Let $c(t)$ be a geodesic as in the statement of the lemma and assume furthermore that both X and Z are nonzero. By differentiating the identity in the lemma and evaluating at $t = 0$ we obtain

$$(*) \ 0 = \langle T J X, X \rangle + \langle T X, J X \rangle = \langle (T J - J T) X, X \rangle$$

since J is skew symmetric. Note that $[T, J] = T J - J T$ is symmetric since T is symmetric and J is skew symmetric. Hence every eigenvalue of $[T, J]$ is zero by $(*)$, which proves that $T J = J T$ and completes the proof of the proposition.

We now prove the lemma. Let $c(t)$ be a geodesic as in the statement of the lemma, and let $J = j(Z)$. By Proposition 2.5.1 we know that $c'(t) = dL_{c(t)}(e^{tJ}X + Z)$ for all t . Hence by the proposition in (1.5) we obtain $\text{Ric}(c'(t), c'(t)) = \mathbf{Ric}(e^{tJ}X + Z, e^{tJ}X + Z) = \mathbf{Ric}(e^{tJ}X, e^{tJ}X) + \mathbf{Ric}(Z, Z) = \langle T e^{tJ}X, e^{tJ}X \rangle - (1/4) \text{trace } J^2$. \square

Proposition 2.5.8. *Let $\mathfrak{N} = V \oplus \mathfrak{Z}$ and suppose that $j(\mathfrak{Z})$ acts irreducibly on V . The Ricci tensors $\mathbf{Ric} : \mathfrak{N} \rightarrow \mathbb{R}$ and $\text{Ric} : TN \rightarrow \mathbb{R}$ are first integrals for the geodesic flow in \mathfrak{N} and TN respectively $\Leftrightarrow T = \sum_{i=1}^p j(Z_i)^2$ is a negative multiple of the identity for any orthonormal basis $\{Z_1, \dots, Z_p\}$ of \mathfrak{Z} .*

Proof. The transformation $T = (1/2) \sum_{i=1}^p j(Z_i)^2$ is negative definite and symmetric, and hence any linear transformation on V that commutes with T must leave invariant each of its eigenspaces. The result now follows immediately from the irreducibility of $j(\mathfrak{Z})$ and the preceding result. \square

Polynomial first integrals of Butler. L. Butler in [Bu] exhibited the following polynomial first integrals for the geodesic flow in \mathfrak{N} . See (6.8b) of [E3] for an independent proof.

Proposition 2.5.9. *(L. Butler) For each integer i with $1 \leq i \leq (1/2) \dim V$, define $\mathbf{f}_i : \mathfrak{N} \rightarrow \mathbb{R}$ by $\mathbf{f}_i(X + Z) = \langle X, j(Z)^{2i}X \rangle$ for all $X \in V$ and $Z \in \mathfrak{Z}$. Then each \mathbf{f}_i is a first integral for the geodesic flow in \mathfrak{N} .*

3. AUTOMORPHISMS AND DERIVATIONS OF 2-STEP NILPOTENT LIE ALGEBRAS

3.1. 2-step nilpotent Lie algebras of type (p,q). A 2-step nilpotent Lie algebra \mathfrak{N} is said to be of type (p, q) if the commutator ideal $[\mathfrak{N}, \mathfrak{N}]$ has dimension p and codimension q , where p and q are positive integers.

Adapted bases and structure spaces. Let \mathfrak{N} be a 2-step nilpotent Lie algebra of type (p, q) . A basis $\mathfrak{B} = \{X_1, \dots, X_q, Z_1, \dots, Z_p\}$ of \mathfrak{N} is said to be *adapted* if $\{Z_1, \dots, Z_p\}$ is a basis of the commutator ideal $[\mathfrak{N}, \mathfrak{N}]$. Adapted bases for \mathfrak{N} always exist since $[\mathfrak{N}, \mathfrak{N}]$ is a subspace of \mathfrak{N} .

Define skew symmetric $q \times q$ matrices C^1, \dots, C^p by the bracket relations $[X_i, X_j] = \sum_{k=1}^p C_{ij}^k Z_k$. It is not difficult to show that the matrices $\{C^1, \dots, C^p\}$ are linearly independent in $\mathfrak{so}(q, \mathbb{R})$; see for example the lemma in the proof of Proposition 2.6 in [E5]. If $W = \text{span} \{C^1, \dots, C^p\} \subseteq \mathfrak{so}(q, \mathbb{R})$, then we call W the *structure space* determined by the adapted basis \mathfrak{B} .

The subspace W in the construction above obviously depends upon the choice of an adapted basis of \mathfrak{N} . However, the set of subspaces W that arise from all possible choices of an adapted basis of \mathfrak{N} is precisely the set $\{gWg^t : g \in GL(p, \mathbb{R})\}$. See Proposition 3.1 of [E4] for details of the proof.

Standard, metric, 2-step nilpotent Lie algebras of type (p,q). We describe a simple family of examples of type (p, q) and then show that every 2-step nilpotent Lie algebra is isomorphic to one of these.

Let $\mathfrak{so}(q, \mathbb{R})$ denote the Lie algebra of $q \times q$ skew symmetric matrices with real entries. Let $\langle \cdot, \cdot \rangle$ denote the positive definite inner product on $\mathfrak{so}(q, \mathbb{R})$ given by $\langle Z, Z' \rangle = -\text{trace}(ZZ')$. We call $\langle \cdot, \cdot \rangle$ the *canonical inner product* on $\mathfrak{so}(q, \mathbb{R})$.

If g is an element of the orthogonal group $O(n, \mathbb{R})$, then conjugation by g on $\mathfrak{so}(q, \mathbb{R})$ preserves the inner product $\langle \cdot, \cdot \rangle$. It is not difficult to show that any positive definite inner product on $\mathfrak{so}(q, \mathbb{R})$ that is invariant under conjugation by elements of $O(n, \mathbb{R})$ is a positive multiple of $\langle \cdot, \cdot \rangle$.

Let W be a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$, and let the vector space $\mathfrak{N} = \mathbb{R}^q \oplus W$ be equipped with the inner product $\langle \cdot, \cdot \rangle^*$ such that $\langle \mathbb{R}^q, W \rangle^* = 0$, $\langle \cdot, \cdot \rangle^* = \langle \cdot, \cdot \rangle$ on W and $\langle \cdot, \cdot \rangle^*$ is the standard inner product on \mathbb{R}^q for which the natural basis $\{e_1, \dots, e_q\}$ is orthonormal. Let $[\cdot, \cdot]$ be the unique bracket structure on \mathfrak{N} such that W lies in the center of \mathfrak{N} and $\langle [X, Y], Z \rangle^* = \langle Z(X), Y \rangle^*$ for arbitrary vectors X, Y in \mathbb{R}^q and Z in W . It is easy to see \mathfrak{N} is a 2-step nilpotent Lie algebra such that $[\mathfrak{N}, \mathfrak{N}] = W$. In particular $\mathfrak{N} = \mathbb{R}^q \oplus W$ is of type (p, q) for every p -dimensional subspace W of $\mathfrak{so}(q, \mathbb{R})$.

We call $\mathfrak{N} = \mathbb{R}^q \oplus W$ a *standard metric 2-step nilpotent Lie algebra of type (p, q)* .

Reduction to standard form.

Proposition 3.1.1. *Let \mathfrak{N} be a 2-step nilpotent Lie algebra of type (p, q) . Then there exists a p -dimensional subspace W of $\mathfrak{so}(q, \mathbb{R})$ such that \mathfrak{N} is isomorphic as a Lie algebra to the standard metric 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$.*

Proof. Let $\mathfrak{B} = \{X_1, \dots, X_q, Z_1, \dots, Z_p\}$ be an adapted basis of \mathfrak{N} , and define skew symmetric $q \times q$ matrices C^1, \dots, C^p as above by the bracket relations $[X_i, X_j] = \sum_{k=1}^p C_{ij}^k Z_k$. Let $\{\rho_1, \dots, \rho_p\}$ be the basis of $W = \text{span}\{C^1, \dots, C^p\}$ such that $\langle C^i, \rho_j \rangle = -\delta_{ij}$, where $\langle \cdot, \cdot \rangle$ is the canonical inner product on $\mathfrak{so}(q, \mathbb{R})$ defined above. Let $T : \mathfrak{N} \rightarrow \mathbb{R}^q \oplus W$ be the linear isomorphism such that $T(X_i) = e_i$ for $1 \leq i \leq q$ and $T(Z_j) = \rho_j$ for $1 \leq j \leq p$. If $\mathfrak{N} = \mathbb{R}^q \oplus W$ is given the standard metric 2-step nilpotent Lie algebra structure above, then it is routine to check that T is also a Lie algebra isomorphism. \square

We may sharpen the result above to show that any metric, 2-step nilpotent Lie algebra can be regarded as a small metric modification of a standard, metric, 2-step nilpotent Lie algebra.

Proposition 3.1.2. *Let $\{\mathfrak{N}^*, \langle \cdot, \cdot \rangle^*\}$ be a metric 2-step nilpotent Lie algebra of type (p, q) . Then there exists a standard, metric, 2-step nilpotent Lie algebra $\{\mathfrak{N} = \mathbb{R}^q \oplus W, \langle \cdot, \cdot \rangle\}$ and an additional inner product $\langle \cdot, \cdot \rangle'$ on \mathfrak{N} such that*

- 1) $\langle \cdot, \cdot \rangle'$ is the standard inner product on \mathbb{R}^q .
- 2) \mathbb{R}^q and W are orthogonal relative to $\langle \cdot, \cdot \rangle'$.
- 3) There exists a linear isometry $T : \{\mathfrak{N}^*, \langle \cdot, \cdot \rangle^*\} \rightarrow \{\mathfrak{N}, \langle \cdot, \cdot \rangle'\}$ that is also a Lie algebra isomorphism.

Proof. Let $\mathfrak{B} = \{X_1, \dots, X_q, Z_1, \dots, Z_p\}$ be an adapted basis of \mathfrak{N}^* that is orthonormal relative to $\langle \cdot, \cdot \rangle^*$. If $\{C_1, \dots, C_p\}, W, \{\rho_1, \dots, \rho_p\}$ and $T : \mathfrak{N}^* \rightarrow \mathbb{R}^q \oplus W$ are defined as in the proof of Proposition 3.1.1, then we know already that T is a Lie algebra isomorphism.

Now let $\langle \cdot, \cdot \rangle'$ be the inner product on $\mathfrak{N} = \mathbb{R}^q \oplus W$ such that $T : \{\mathfrak{N}^*, \langle \cdot, \cdot \rangle^*\} \rightarrow \{\mathfrak{N}, \langle \cdot, \cdot \rangle'\}$ is a linear isometry. If $V = \text{span}\{X_1, \dots, X_q\}$ and $\mathfrak{W} = \text{span}\{Z_1, \dots, Z_p\}$, then $\mathbb{R}^q = T(V)$ and $W = T(\mathfrak{W})$ are orthogonal relative to $\langle \cdot, \cdot \rangle'$. Moreover, $\{e_1, \dots, e_q\}$ is an orthonormal basis of \mathbb{R}^q relative to $\langle \cdot, \cdot \rangle'$ since $\{X_1, \dots, X_q\}$ is an orthonormal basis of V relative to $\langle \cdot, \cdot \rangle^*$. Hence $\langle \cdot, \cdot \rangle'$ agrees with the standard inner product on \mathbb{R}^q , and the proof is complete. \square

Isomorphism classes. The next result, which is Corollary 3.5c of [E4], gives a necessary and sufficient condition for two standard, metric, 2-step nilpotent Lie algebras of type (p, q) to be isomorphic.

Proposition 3.1.3. *Let W_1 and W_2 be p -dimensional subspaces of $\mathfrak{so}(q, \mathbb{R})$, and let $\mathfrak{N}_1 = \mathbb{R}^q \oplus W_1$ and $\mathfrak{N}_2 = \mathbb{R}^q \oplus W_2$ be the corresponding standard, metric 2-step nilpotent Lie algebras. Then \mathfrak{N}_1 is isomorphic to $\mathfrak{N}_2 \Leftrightarrow$ there exists $g \in GL(q, \mathbb{R})$ such that $W_2 = g(W_1) = gW_1g^t$.*

For integers $p \geq 2$ and $q \geq 3$ we let $G(p, \mathfrak{so}(q, \mathbb{R}))$ denote the Grassmann manifold of p -dimensional subspaces of $\mathfrak{so}(q, \mathbb{R})$. For later use we compare the orbits of $\overline{G} = GL(q, \mathbb{R})$ and $G = SL(q, \mathbb{R})$ on $G(p, \mathfrak{so}(q, \mathbb{R}))$.

Proposition 3.1.4. *Fix an element α of \overline{G} with $\det \alpha < 0$. For any $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$ we have $\overline{G}(W) = G(W) \cup G(\alpha(W))$*

Proof. Let $g \in \overline{G}$. If $\det g > 0$, then $g = \lambda h$ for $\lambda \in (0, \infty)$ and $h \in G$. It follows that $g(W) = h(W)$. If $\det g < 0$, then $\det(g\alpha^{-1}) > 0$ and $g = \lambda h\alpha$ for $\lambda \in (0, \infty)$ and $h \in G$. In this case $g(W) = h(\alpha(W))$. \square

Corollary 3.1.5. *Let $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$. Then $\overline{G}(W) = G(W) \Leftrightarrow$ there exists $\alpha \in \overline{G}$ with $\det \alpha < 0$ such that $\alpha(W) = W$. In particular, $\overline{G}(W) = G(W)$ for all $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$ if q is odd.*

Proof. Let $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$ be given. If there exists $\alpha \in \overline{G}$ with $\det \alpha < 0$ such that $\alpha(W) = W$, then $\overline{G}(W) = G(W)$ by Proposition 3.1.4. Now suppose that $\overline{G}(W) = G(W)$. Let $g \in \overline{G}$ be given with $\det g < 0$, and choose $h \in G$ so that $g(W) = h(W)$. If $\alpha = h^{-1}g$, then $\alpha(W) = W$ and $\det \alpha < 0$.

If q is odd and $\alpha = -\text{Id}$, then $\det \alpha < 0$ and $\alpha(W) = W$ for all $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$. This completes the proof of the corollary. \square

3.2. The actions of $GL(q, \mathbb{R})$ and $\text{End}(\mathbb{R}^q)$ on $\mathfrak{so}(q, \mathbb{R})$. If $g \in GL(q, \mathbb{R})$, then g acts as an invertible linear transformation $\rho(g)$ on $\mathfrak{so}(q, \mathbb{R})$ by $\rho(g)(Z) = gZg^t$ for all $Z \in \mathfrak{so}(q, \mathbb{R})$. The map $\rho : GL(q, \mathbb{R}) \rightarrow GL(\mathfrak{so}(q, \mathbb{R}))$ is a group homomorphism whose kernel consists of $\{\pm \text{Id}\}$ for $q \geq 3$.

If $X \in \text{End}(\mathbb{R}^q)$, the Lie algebra of $GL(q, \mathbb{R})$, then we get a corresponding action on $\mathfrak{so}(q, \mathbb{R})$ by defining $d\rho(X)(Z) = XZ + ZX^t$ for all $X \in \text{End}(\mathbb{R}^q)$ and all $Z \in \mathfrak{so}(q, \mathbb{R})$. The map $d\rho : \text{End}(\mathbb{R}^q) \rightarrow \text{End}(\mathfrak{so}(q, \mathbb{R}))$ is a Lie algebra homomorphism that is injective for $q \geq 3$.

In the sequel we will typically suppress the notations ρ and $d\rho$ unless the context is unclear.

Some of the metric properties of elements g or X acting on \mathbb{R}^q are preserved by the elements $\rho(g)$ or $d\rho(X)$ acting on $\mathfrak{so}(q, \mathbb{R})$.

Proposition. *Let $g \in GL(q, \mathbb{R})$ and $X \in \text{End}(\mathbb{R}^q)$ be any elements. Let g^t or X^t denote the metric transpose of g or X acting on \mathbb{R}^q with the standard inner product. Let $\rho(g)^t$ or $d\rho(X)^t$ denote the metric transpose of $\rho(g)$ or $d\rho(X)$ acting on $\mathfrak{so}(q, \mathbb{R})$ with respect to the canonical inner product \langle, \rangle such that $\langle Z, Z' \rangle = -\text{trace}(ZZ')$. Then*

1) $\rho(g^t) = \rho(g)^t$ and $d\rho(X^t) = d\rho(X)^t$; that is, $\langle \rho(g)(Z), Z' \rangle = \langle Z, \rho(g^t)(Z') \rangle$ and $\langle d\rho(X)(Z), Z' \rangle = \langle Z, d\rho(X^t)(Z') \rangle$ for all $Z, Z' \in \mathfrak{so}(q, \mathbb{R})$.

2) If h is a positive definite, symmetric transformation on \mathbb{R}^q , then the extensions $\rho(h)$ and $d\rho(h)$ to $\mathfrak{so}(q, \mathbb{R})$ are positive definite and symmetric.

Proof. We omit the proof of 1), which follows routinely.

If h is a symmetric linear transformation on \mathbb{R}^q , then its extensions $\rho(h)$ and $d\rho(h)$ to $\mathfrak{so}(q, \mathbb{R})$ are symmetric by 1). If h is a diagonal matrix with positive entries, then it is routine to show that $\langle \rho(h)(Z), Z \rangle$ is positive for any nonzero element Z of $\mathfrak{so}(q, \mathbb{R})$. If h is an arbitrary positive definite symmetric transformation on \mathbb{R}^q , then there exist an element g in $O(q, \mathbb{R})$ and a diagonal matrix h' with positive entries such that $h = g^{-1}h'g = g^t h' g$. If Z is any nonzero element of $\mathfrak{so}(q, \mathbb{R})$, then $\langle \rho(h)(Z), Z \rangle = \langle \rho(g^t h' g)(Z), Z \rangle = \langle \rho(h')(\rho(g)(Z)), \rho(g)(Z) \rangle$, which is positive as noted above. A similar argument shows that $\langle d\rho(h)(Z), Z \rangle$ is positive for any nonzero element Z of $\mathfrak{so}(q, \mathbb{R})$. \square

3.3. Free 2-step nilpotent Lie algebras.

Universal mapping definition. If we choose $W = \mathfrak{so}(q, \mathbb{R})$ in the construction above, then the metric 2-step nilpotent Lie algebra $\mathfrak{F}_2(q) = \mathbb{R}^q \oplus \mathfrak{so}(q, \mathbb{R})$ has the following universal property, which characterizes it up to isomorphism : Let \mathfrak{N} be any 2-step nilpotent Lie algebra, and let $\{y_1, \dots, y_q\}$ be any q points in \mathfrak{N} , not necessarily distinct. Let $\{x_1, \dots, x_q\}$ be any basis of \mathbb{R}^q . Then there exists a unique Lie algebra homomorphism $T : \mathfrak{F}_2(q) \rightarrow \mathfrak{N}$ such that $T(x_i) = y_i$ for $1 \leq i \leq q$. See (2.3) of [E4] for details. We call $\mathfrak{F}_2(q)$ the *free 2 – step nilpotent Lie algebra on q generators*. Clearly $\mathfrak{F}_2(q)$ has type (D, q) , where $D = (1/2)q(q-1)$.

The definition of $\mathfrak{F}_2(q)$ above is the most convenient for our purposes, but it is not the most intrinsic since it comes with a preferred inner product. We may also define the bracket $[\ , \]$ directly on $\mathfrak{F}_2(q) = \mathbb{R}^q \oplus \mathfrak{so}(q, \mathbb{R})$ by requiring that $\mathfrak{so}(q, \mathbb{R})$ lie in the center of $\mathfrak{F}_2(q)$ and $[x, y] = - (1/2) \{xy^t - yx^t\}$ for all $x, y \in \mathbb{R}^q$, where x, y are regarded as $q \times 1$ column vectors.

Alternate definition of $\mathfrak{F}_2(q)$. The map $\phi : \mathbb{R}^q \times \mathbb{R}^q \rightarrow \mathfrak{so}(q, \mathbb{R})$ given by $\phi(x, y) = - (1/2) \{xy^t - yx^t\}$ is an alternating, bilinear map, and it induces an isomorphism $T : \Lambda^2(\mathbb{R}^q) \rightarrow \mathfrak{so}(q, \mathbb{R})$ such that $T(x \wedge y) = - (1/2) \{xy^t - yx^t\}$ for all x, y in \mathbb{R}^q . Define a 2-step nilpotent Lie algebra $\mathfrak{F}_2(q)^* = \mathbb{R}^q \oplus \Lambda^2(\mathbb{R}^q)$ such that $\Lambda^2(\mathbb{R}^q)$ lies in the center of $\mathfrak{F}_2(q)^*$ and $[x, y] = x \wedge y$ for all x, y in \mathbb{R}^q . It is not difficult to show that the map $F : \mathfrak{F}_2(q)^* \rightarrow \mathfrak{F}_2(q)$ given by $F(x, Z) = (x, T(Z))$ is a Lie algebra isomorphism.

Automorphisms and derivations. The result below is proved in (2.5) of [E4].

Proposition. *Let $\mathfrak{F}_2(q) = \mathbb{R}^q \oplus \mathfrak{so}(q, \mathbb{R})$ be the free 2-step nilpotent Lie algebra on q generators. For every automorphism ϕ of $\mathfrak{F}_2(q)$ there exist unique elements g of $GL(q, \mathbb{R})$ and S of $Hom(\mathbb{R}^q, \mathfrak{so}(q, \mathbb{R}))$ such that*

$$\begin{aligned} a) \phi(v) &= g(v) + S(v) && \text{for all } v \in \mathbb{R}^q. \\ b) \phi(Z) &= gZg^t && \text{for all } Z \in \mathfrak{so}(q, \mathbb{R}). \end{aligned}$$

Conversely, given $(g, S) \in GL(q, \mathbb{R}) \times Hom(\mathbb{R}^q, \mathfrak{so}(q, \mathbb{R}))$ there exists a unique automorphism ϕ of $\mathfrak{F}_2(q)$ such that a) and b) are satisfied.

If \mathfrak{H} is a Lie algebra, then an element $A \in End(\mathfrak{H})$ is a *derivation* of \mathfrak{H} if $A[x, y] = [Ax, y] + [x, Ay]$ for all $x, y \in \mathfrak{H}$. The set $Der(\mathfrak{H})$ of derivations of \mathfrak{H} is the Lie algebra of $Aut(\mathfrak{H})$. From the result above we now obtain the following

Corollary. *Let $\mathfrak{F}_2(q) = \mathbb{R}^q \oplus \mathfrak{so}(q, \mathbb{R})$ be the free 2-step nilpotent Lie algebra on q generators. For every derivation D of $\mathfrak{F}_2(q)$ there exist unique elements A of $End(\mathbb{R}^q)$ and B of $Hom(\mathbb{R}^q, \mathfrak{so}(q, \mathbb{R}))$ such that*

$$\begin{aligned} a) D(v) &= A(v) + B(v) && \text{for all } v \in \mathbb{R}^q. \\ b) D(Z) &= AZ + ZA^t && \text{for all } Z \in \mathfrak{so}(q, \mathbb{R}). \end{aligned}$$

Conversely, given $(A, B) \in End(\mathbb{R}^q) \times Hom(\mathbb{R}^q, \mathfrak{so}(q, \mathbb{R}))$ there exists a unique derivation D of $\mathfrak{F}_2(q)$ such that a) and b) are satisfied.

3.4. Automorphisms and derivations of arbitrary 2-step nilpotent Lie algebras. Let $\mathfrak{F}_2(q) = \mathbb{R}^q \oplus \mathfrak{so}(q, \mathbb{R})$ be the free 2-step nilpotent Lie algebra on q generators. Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be a standard, metric, 2-step nilpotent Lie algebra of type (p, q) , where W is a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$. Let $\pi : \mathfrak{F}_2(q) \rightarrow \mathfrak{N}$ be the unique Lie algebra homomorphism such that $\pi(e_i) = e_i$ for $1 \leq i \leq q$. It is not difficult to show that π is surjective, π is the identity on \mathbb{R}^q and $Ker \pi = W^\perp$, the orthogonal complement of W in $\mathfrak{so}(q, \mathbb{R})$ relative to the canonical inner product $\langle \ , \ \rangle$. Hence \mathfrak{N} is isomorphic to $\mathfrak{F}_2(q)/W^\perp$. See (3.4) of [E4] for details.

Let $\varphi : \mathfrak{N} = \mathbb{R}^q \oplus W \rightarrow \mathfrak{N}$ be any Lie algebra automorphism, and let $\{v_1, \dots, v_q\}$ be any vectors in $\mathfrak{F}_2(q) = \mathbb{R}^q \oplus \mathfrak{so}(q, \mathbb{R})$ such that $\pi(v_i) = \varphi(e_i)$ for $1 \leq i \leq q$. Then there exists a unique automorphism $\varphi' : \mathfrak{F}_2(q) \rightarrow \mathfrak{F}_2(q)$ such that $\varphi'(e_i) = v_i$ for $1 \leq i \leq q$ and $\pi \circ \varphi' = \varphi \circ \pi$. (See for example Proposition 2.2d of [E4]). It follows immediately that φ' leaves invariant $\text{Ker } \pi = W^\perp$. Conversely, if φ' is an automorphism of $\mathfrak{F}_2(q)$ that leaves invariant W^\perp , then there exists a unique automorphism φ of $\mathfrak{N} = \mathbb{R}^q \oplus W$ such that $\pi \circ \varphi' = \varphi \circ \pi$. It is easy to see that φ is the identity on $\mathfrak{N} \Leftrightarrow \varphi'$ has the form (Id, S) in the notation of (3.3), where $S \in \text{Hom}(\mathbb{R}^q, W^\perp)$. We summarize this discussion as follows.

Proposition 3.4.1. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be a standard, metric, 2-step nilpotent Lie algebra of type (p, q) , where W is a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$. Let $G'_W = \{\varphi' \in \text{Aut}(\mathfrak{F}_2(q)) : \varphi'(W^\perp) = W^\perp\}$ and let $H'_W = \{\varphi' \in \text{Aut}(\mathfrak{F}_2(q)) :$*

$$\begin{aligned} a) \varphi'(v) &= v + S(v) && \text{for all } v \in \mathbb{R}^q \text{ and some } S \in \text{Hom}(\mathbb{R}^q, W^\perp) \\ b) \varphi' &\equiv \text{Id on } \mathfrak{so}(q, \mathbb{R}^q)\}. \end{aligned}$$

Let $F : G'_W \rightarrow \text{Aut}(\mathfrak{N})$ be the unique map such that $\pi \circ \varphi' = F(\varphi') \circ \pi$. Then

- 1) *F is a surjective homomorphism with kernel H'_W .*
- 2) *H'_W , may be bijectively identified with $\text{Hom}(\mathbb{R}^q, W^\perp)$.*

Reduction to $\text{SL}(q, \mathbb{R})$. As in (3.1) we let \overline{G} and G denote $\text{GL}(q, \mathbb{R})$ and $\text{SL}(q, \mathbb{R})$ respectively. We let $\overline{G}_W = \{g \in \overline{G} : g(Z) = gZg^t \in W \text{ for all } Z \in W\}$ and we let $G_W = \{g \in G : g(Z) = gZg^t \in W \text{ for all } Z \in W\}$. We let $\overline{\mathfrak{G}}, \overline{\mathfrak{G}}_W, \mathfrak{G}$ and \mathfrak{G}_W denote the corresponding Lie algebras. In particular, \mathfrak{G} and \mathfrak{G}_W are the subalgebras of $\overline{\mathfrak{G}}$ and $\overline{\mathfrak{G}}_W$ consisting of those elements with trace zero.

The groups \overline{G}_W and \overline{G}_{W^\perp} are isomorphic under the involution $\sigma : g \rightarrow (g^t)^{-1}$. Clearly $\sigma(G_W) = G_{W^\perp}$ also. One can show that the groups \overline{G}_W and G_W have finitely many connected components. In general for all W in a dense, open subset O of $G(p, \mathfrak{so}(q, \mathbb{R}))$ the group G_W is finite and $(\overline{G}_W)_O$ consists of all positive multiples of the identity. See (4.3) below for a more precise statement.

The group \overline{G}_W has a 1-dimensional central subgroup isomorphic to $\mathbb{R}^* = \mathbb{R} - \{0\}$ that consists of the nonzero multiples of the identity. Similarly the set of multiples of the identity is a 1-dimensional central subspace of $\overline{\mathfrak{G}}_W$. Clearly $\overline{G}_W = \mathbb{R}^* \cdot G_W$ (direct product) and $\overline{\mathfrak{G}}_W = \mathbb{R} \oplus \mathfrak{G}_W$ (direct sum).

It is usually more convenient to work with $G_W \subseteq \text{SL}(q, \mathbb{R})$ and $\mathfrak{G}_W \subseteq \mathfrak{sl}(q, \mathbb{R})$ rather than with \overline{G}_W and $\overline{\mathfrak{G}}_W$, and no information is lost in doing so. This will frequently be our point of view. As an example we observe that \mathfrak{G}_W is self adjoint $\Leftrightarrow \overline{\mathfrak{G}}_W$ is self adjoint.

Canonical decomposition of $\text{Aut}(\mathfrak{N})$. Let $\mathfrak{N} = \mathbb{R}^q \oplus W$. It is useful to have a more intrinsic description of $\text{Aut}(\mathfrak{N})$ than that given by Proposition 3.4.1. We show that $\text{Aut}(\mathfrak{N})$ is a semidirect product $U \cdot \overline{H}$, where U is a normal, abelian, unipotent subgroup of $\text{Aut}(\mathfrak{N})$ and \overline{H} is a subgroup of $\text{Aut}(\mathfrak{N})$ isomorphic to \overline{G}_{W^\perp} .

The unipotent subgroup U . For every element $A \in \text{Hom}(\mathbb{R}^q, W)$ we define $u_A \in \text{End}(\mathfrak{N})$ by $u_A(v) = v + A(v)$ for $v \in \mathbb{R}^q$ and $u_A(Z) = Z$ for all $Z \in W$. It is not difficult to see that $u_A \in \text{Aut}(\mathfrak{N})$ and $u_A \circ u_B = u_B \circ u_A = u_{A+B}$ for all $A, B \in \text{Hom}(\mathbb{R}^q, W)$. Moreover, if \mathfrak{B} is a basis of \mathfrak{N} that is a union of bases from \mathbb{R}^q and W , then it is evident that each element u_A has a lower triangular matrix with 1's on the diagonal relative to the basis \mathfrak{B} . Hence, if $U = \{u_A : A \in \text{Hom}(\mathbb{R}^q, W)\}$, then U is an abelian unipotent subgroup of $\text{Aut}(\mathfrak{N})$.

The map $T : \text{End}(\mathbb{R}^q) \rightarrow \text{End}(\mathfrak{N})$. For $g \in \text{End}(\mathbb{R}^q)$ define $T(g) \in \text{End}(\mathfrak{N})$ by $T(g)(v) = g(v)$ for all $v \in \mathbb{R}^q$ and $T(g)(Z) = \pi(gZg^t)$ for all $Z \in W$, where $\pi : \mathfrak{F}_2(q) \rightarrow \mathfrak{N}$ is the projection homomorphism with $\text{Ker } \pi = W^\perp$. The map T is one-one, but the restriction $T : \text{GL}(\mathbb{R}^q) \rightarrow \text{GL}(\mathfrak{N})$ is not a group homomorphism. However, we have the following :

Lemma. *The map $T : \overline{G}_{W^\perp} \rightarrow GL(\mathfrak{N})$ is a group homomorphism whose image lies in $\text{Aut}(\mathfrak{N})$. For $g \in \overline{G}$ the element $T(g)$ lies in $\text{Aut}(\mathfrak{N}) \Leftrightarrow g \in \overline{G}_{W^\perp}$.*

Proof. Let $g \in \overline{G}_{W^\perp}$ be given and define $\tilde{T}(g) \in \text{Aut}(\mathfrak{F}_2(q))$ by $\tilde{T}(g)(v) = g(v)$ for all $v \in \mathbb{R}^q$ and $\tilde{T}(g)(Z) = gZg^t$ for all $Z \in \mathfrak{so}(q, \mathbb{R})$. Since $\tilde{T}(g)$ leaves invariant $W^\perp = \text{Ker } \pi$ it follows that there exists an element $\varphi \in \text{Aut}(\mathfrak{N})$ such that $\varphi \circ \pi = \pi \circ \tilde{T}(g)$. From the definitions it follows immediately that $\varphi = T(g)$. Since $\tilde{T} : GL(q, \mathbb{R}) \rightarrow \text{Aut}(\mathfrak{F}_2(q))$ is a homomorphism we conclude that $T : \overline{G}_{W^\perp} \rightarrow \text{Aut}(\mathfrak{N})$ is a homomorphism.

The discussion above shows that $T(g) \in \text{Aut}(\mathfrak{N})$ if $g \in \overline{G}_{W^\perp}$. We omit the proof that $g \in \overline{G}_{W^\perp}$ if $T(g) \in \text{Aut}(\mathfrak{N})$. \square

Proposition 3.4.2. *Let $\overline{H} = T(\overline{G}_{W^\perp})$ and let $U = \{u_A : A \in \text{Hom}(\mathbb{R}^q, W)\}$. Then $U \cap \overline{H} = \{1\}$, U is a normal, abelian unipotent subgroup of $\text{Aut}(\mathfrak{N})$ and $\text{Aut}(\mathfrak{N})$ is the semidirect product $U \cdot \overline{H}$.*

Proof. It is evident that $U \cap \overline{H} = \{1\}$, and we already observed that U is abelian and unipotent. It is routine to show that \overline{H} normalizes U . Hence it remains only to show that every element φ of $\text{Aut}(\mathfrak{N})$ can be written uniquely as $u_A \circ T(g)$ for suitable elements $A \in \text{Hom}(\mathbb{R}^q, W)$ and $g \in \overline{G}_{W^\perp}$. The uniqueness of this expression is easy to show and we only prove the existence. Given $\varphi \in \text{Aut}(\mathfrak{N})$ it follows from the discussion at the beginning of (3.4) that there exists an element $\tilde{\varphi}$ of $\text{Aut}(\mathfrak{F}_2(q))$ such that $\varphi \circ \pi = \pi \circ \tilde{\varphi}$. By the proposition in (3.3) we may choose elements $g \in GL(q, \mathbb{R})$ and $S \in \text{Hom}(\mathbb{R}^q, \mathfrak{so}(q, \mathbb{R}))$ such that $\tilde{\varphi}(v) = g(v) + S(v)$ for all $v \in \mathbb{R}^q$ and $\tilde{\varphi}(Z) = gZg^t$ for all $Z \in \mathfrak{so}(q, \mathbb{R})$. It follows from the condition $\varphi \circ \pi = \pi \circ \tilde{\varphi}$ that g must leave invariant $\text{Ker } \pi = W^\perp$; i.e. $g \in \overline{G}_{W^\perp}$. For $v \in \mathbb{R}^q$ we have $\varphi(v) = (\varphi \circ \pi)(v) = (\pi \circ \tilde{\varphi})(v) = \pi(g(v) + S(v)) = g(v) + A(g(v))$, where $A = \pi \circ S \circ g^{-1} \in \text{Hom}(\mathbb{R}^q, W)$. Similarly, for $Z \in W$ we have $\varphi(Z) = (\varphi \circ \pi)(Z) = (\pi \circ \tilde{\varphi})(Z) = \pi(gZg^t)$. It follows that $\varphi = u_A \circ T(g)$ since these two elements of $\text{Aut}(\mathfrak{N})$ have the same values on \mathbb{R}^q and W . \square

Remark. We note that $\overline{H} = \{\varphi \in \text{Aut}(\mathfrak{N}) : \varphi(\mathbb{R}^q) = \mathbb{R}^q\}$. This follows easily from the proposition above, and we omit the details.

Maximal compact subgroups of $\text{Aut}(\mathfrak{N})_o$. Because U is unipotent it contains no compact subgroups except $\{1\}$. It follows from the result above that every maximal compact subgroup K of $\text{Aut}(\mathfrak{N})_o$ is conjugate by an element of U to a maximal compact subgroup of \overline{H}_o . Since $T : (\overline{G}_{W^\perp})_o \rightarrow \overline{H}_o$ is an isomorphism the maximal compact subgroups of \overline{H}_o are isomorphic to the maximal compact subgroups of $(\overline{G}_{W^\perp})_o$. We omit the details.

Aut(\mathfrak{N}) in the generic case. Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be a standard, metric, 2-step nilpotent Lie algebra of type (p, q) . With a few exceptions, as noted below in (4.3), there exists a dense open subset O of $G(p, \mathfrak{so}(q, \mathbb{R}))$ such that if W lies in O then $\mathfrak{G}_W = \{0\}$. For $W \in O$ the group G_W is finite since G_W is algebraic and therefore has only finitely many connected components. It follows that G_{W^\perp} is finite since G_{W^\perp} is isomorphic to G_W under the map that sends g to $(g^t)^{-1}$. In particular, by the discussion above, the identity component \overline{H}_o of \overline{H} consists of the positive multiples of the identity.

Now let $W \in O$, and let φ be an automorphism of the standard, metric, 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$ of type (p, q) . Since $\text{Aut}(\mathfrak{N})$ has finitely many components it follows that $\varphi^N \in \text{Aut}(\mathfrak{N})_o = U \cdot \overline{H}_o$ for some positive integer N . Write $\varphi^N = u \cdot \lambda \text{Id}$, where $u \in U$ and $\lambda \in (0, \infty)$. If $\psi = \lambda \text{Id} \in \overline{H}_o \subseteq \text{End}(\mathfrak{N})$, then $\psi = \lambda \text{Id}$ on \mathbb{R}^q and $\psi = \lambda^2 \text{Id}$ on W . It follows that $\det(\psi) = \lambda^{q+2p}$. If $\det(\varphi) = \pm 1$, then $\lambda = 1$ since u is unipotent and λ is positive. Note that the eigenvalues of φ^N are N^{th} powers of the eigenvalues of φ . We have proved

Proposition 3.4.3. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be a standard, metric, 2-step nilpotent Lie algebra of type (p, q) , and assume that (p, q) does not belong to the list in (4.3). Then there exists a dense open subset O of $G(p, \mathfrak{so}(q, \mathbb{R}))$ such that if W lies in O then every element of $\text{Aut}(\mathfrak{N})$ with determinant ± 1 has all eigenvalues of modulus 1.*

Canonical decomposition of $\text{Der}(\mathfrak{N})$. We recall from (3.2) that an element A of $\text{End}(\mathbb{R}^q)$ extends to an element of $\text{End}(\mathfrak{so}(q, \mathbb{R}))$ by defining $A(Z) = AZ + ZA^t$ for all $Z \in \mathfrak{so}(q, \mathbb{R})$.

Proposition 3.4.4. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be a standard, metric, 2-step nilpotent Lie algebra of type (p, q) . Let $\pi : \mathfrak{F}_2(q) \rightarrow \mathfrak{N}$ denote the orthogonal projection homomorphism with kernel W^\perp . Let D be a derivation of \mathfrak{N} . Then there exist unique elements A of $\text{End}(\mathbb{R}^q)$ and B of $\text{Hom}(\mathbb{R}^q, W)$ such that*

- 1) $A(W^\perp) \subseteq W^\perp$.
- 2) $D(v) = A(v) + B(v)$ for all $v \in \mathbb{R}^q$.
- $D(Z) = \pi(AZ + ZA^t)$ for all $Z \in W$.

Proof. If \tilde{D} is a derivation of $\mathfrak{F}_2(q)$ that leaves invariant $W^\perp = \text{Ker } \pi$, then there exists a unique element D of $\text{End}(\mathfrak{N})$ such that $D \circ \pi = \pi \circ \tilde{D}$. It follows routinely that D is a derivation of \mathfrak{N} . Let \mathcal{D} denote the vector space of all derivations of \mathfrak{N} that arise in this manner. It is easy to check that \mathcal{D} consists of the elements of $\text{End}(\mathfrak{N})$ that are described above. If $\overline{\mathfrak{G}}_{W^\perp} = \{A \in \text{End}(\mathbb{R}^q) : A(W^\perp) \subseteq W^\perp\}$, then $\overline{\mathfrak{G}}_{W^\perp}$ is the Lie algebra of \overline{G}_{W^\perp} . Hence from Proposition 3.4.2 we obtain $\dim \mathcal{D} = \dim \overline{\mathfrak{G}}_{W^\perp} + \dim \text{Hom}(\mathbb{R}^q, W) = \dim \overline{G}_{W^\perp} + \dim \text{Hom}(\mathbb{R}^q, W) = \dim \text{Aut}(\mathfrak{N}) = \dim \text{Der}(\mathfrak{N})$. It follows that $\mathcal{D} = \text{Der}(\mathfrak{N})$. \square

It will be useful to rephrase the result above in a way that is parallel to proposition 3.4.2. Define a Lie algebra homomorphism $dT : \overline{\mathfrak{G}}_{W^\perp} \rightarrow \text{Der}(\mathfrak{N})$ as follows : for $A \in \overline{\mathfrak{G}}_{W^\perp}$ define $dT(A)(v) = A(v)$ for all $v \in \mathbb{R}^q$ and $dT(A)(Z) = \pi(AZ + ZA^t)$ for all $Z \in W$. Let $\overline{\mathfrak{H}} = dT(\overline{\mathfrak{G}}_{W^\perp})$.

For each element $B \in \text{Hom}(\mathbb{R}^q, W)$ define $u_B \in \text{Der}(\mathfrak{N})$ by $u_B(v) = B(v)$ for all $v \in \mathbb{R}^q$ and $u_B(Z) = 0$ for all $Z \in W$. If $\mathfrak{U} = \{u_B : B \in \text{Hom}(\mathbb{R}^q, W)\}$, then $u_1 \circ u_2 = 0$ for all elements $u_1, u_2 \in \mathfrak{U}$. Clearly $\mathfrak{U} \subseteq \text{Der}(\mathfrak{N})$ by the proposition above, and it is easy to see that \mathfrak{U} is the Lie algebra of the normal, abelian unipotent subgroup U of Proposition 3.4.2. Hence we obtain

Proposition 3.4.5. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$. Then $\text{Der}(\mathfrak{N}) = \overline{\mathfrak{H}} \oplus \mathfrak{U}$, where \mathfrak{U} is an abelian, nilpotent ideal of $\text{Der}(\mathfrak{N})$ and $\overline{\mathfrak{H}}$ is a subalgebra of $\text{Der}(\mathfrak{N})$ that is isomorphic to $\overline{\mathfrak{G}}_{W^\perp}$.*

3.5. Anosov diffeomorphisms. The study of Anosov diffeomorphisms on tori has received considerable attention over the years, and it is a natural idea to extend this research by considering Anosov diffeomorphisms on compact nilmanifolds. Let N be a simply connected, nilpotent Lie group that admits a lattice Γ . By analogy with tori one looks for automorphisms of N that leave invariant the lattice Γ , and one then obtains an induced diffeomorphism on the compact quotient $\Gamma \backslash N$. Such diffeomorphisms of $\Gamma \backslash N$ are called *algebraic*, and the goal is to find algebraic Anosov diffeomorphisms. A result of Manning [Man] says that any Anosov diffeomorphism on $\Gamma \backslash N$ is topologically conjugate to an algebraic Anosov diffeomorphism. Moreover, if Γ_1 and Γ_2 are lattices in N such that $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2 , then $\Gamma_1 \backslash N$ admits an Anosov diffeomorphism $\Leftrightarrow \Gamma_2 \backslash N$ admits an Anosov diffeomorphism. See for example [Da]. Hence by the discussion in (1.2) one may restrict attention to full lattices Γ in N , namely, those for which $\log \Gamma$ is a vector lattice in the Lie algebra \mathfrak{N} . To find automorphisms ϕ of N that leave Γ invariant reduces to the problem of finding automorphisms $d\phi$ of \mathfrak{N} that leave $\log \Gamma$ invariant.

Let \mathfrak{B} be a basis for \mathfrak{N} that is also a \mathbb{Z} -basis for $\log \Gamma$. One can show that the structure constants of \mathfrak{B} lie in \mathbb{Z} . The problem of finding an algebraic Anosov diffeomorphism of $\Gamma \backslash N$ thus becomes

the problem of finding an automorphism $d\phi$ of \mathfrak{N} with no eigenvalues of modulus 1 whose matrix relative to \mathfrak{B} has integer entries and determinant 1 or -1 .

This apparently simple reformulation of the Anosov diffeomorphism problem for compact nilmanifolds conceals its difficulty. Although there are a number of examples of Anosov diffeomorphisms on compact nilmanifolds that have been found by various techniques, there is still no comprehensive theory. See [La1] for a brief history of the efforts on this problem and also the articles [Da], [DeK], [DeKM] and [Malf 1,2].

Existence results. In the context of nilmanifolds, one must begin with a Lie algebra \mathfrak{N} that admits a basis \mathfrak{B} with rational structure constants in order to obtain a lattice Γ in the simply connected nilpotent Lie group N with Lie algebra \mathfrak{N} . These Lie algebras \mathfrak{N} are already rare in a probabilistic sense as explained in (5.3) below, even for 2-step nilpotent Lie algebras. However, there are still plenty of examples.

Let \mathfrak{N} be a 2-step nilpotent Lie algebra that admits a basis \mathfrak{B} with rational structure constants. A subspace W of \mathfrak{N} is said to be *rational* if it has a basis in \mathbb{Q} -span(\mathfrak{B}). A rational Lie algebra \mathfrak{N} is said to be *graded* if it can be written as a direct sum $\mathfrak{N} = \mathfrak{N}_1 \oplus \dots \oplus \mathfrak{N}_k$ of rational subspaces $\{\mathfrak{N}_i\}$ such that $[\mathfrak{N}_i, \mathfrak{N}_j] \subseteq \mathfrak{N}_{i+j}$ for all i, j . It is not difficult to see that any rational 2-step nilpotent Lie algebra \mathfrak{N} is graded if one sets $\mathfrak{N}_2 = [\mathfrak{N}, \mathfrak{N}]$ and chooses \mathfrak{N}_1 to be a suitable vector space complement of \mathfrak{N}_2 in \mathfrak{N} .

Recently J. Lauret in [La2] proved the following result :

Theorem 3.5.1. *Let \mathfrak{N} be a graded nilpotent Lie algebra. Then the direct sum $\mathfrak{N} \oplus \mathfrak{N}$ admits an Anosov automorphism.*

The proof is elementary, but the result has powerful consequences that settle several previously open problems. See [La2] for further discussion.

Nonexistence results in the 2-step nilpotent case.

Nonexistence results in the generic 2-step nilpotent case. For a generic 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$ of type (p, q) we showed in Proposition 3.4.3 that if (p, q) does not belong to a small list found in (4.3) then every automorphism of \mathfrak{N} with determinant ± 1 has all eigenvalues of modulus 1. In particular \mathfrak{N} admits no Anosov automorphisms in this case.

Even in the generic case one can easily find examples of 2-step nilpotent Lie algebras $\mathfrak{N} = \mathbb{R}^q \oplus W$ with rational structures. Simply choose W to be a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$ that is spanned by elements of $\mathfrak{so}(q, \mathbb{R})$ that have rational entries.

Nonexistence results in the Lie triple system examples. It is natural to look for Anosov automorphisms in standard, metric, 2-step nilpotent Lie algebras $\mathfrak{N} = \mathbb{R}^q \oplus W$ where W is a Lie triple system with compact center. These Lie algebras are known to admit rational structures by Proposition 4.3c of [E5], and they contain a lot of algebraic structure that one would expect to improve the odds of success. Although the situation is not yet clear in this context there are some partial results, including nonexistence results. We present one of these.

Proposition 3.5.2. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be a standard, metric, 2-step nilpotent Lie algebra such that W is a subspace of $\mathfrak{so}(q, \mathbb{R})$ that satisfies one of the following conditions :*

1) *W is of Clifford type ; that is, Z^2 is a negative multiple of the identity for every nonzero element Z of W .*

2) *$W = \mathfrak{so}(3, \mathbb{R})$ and q is even.*

If ϕ is any automorphism of \mathfrak{N} with $\det(\phi) = \pm 1$, then $\phi(W) \subseteq W$ and all eigenvalues of ϕ on W have modulus 1. In particular, \mathfrak{N} admits no Anosov automorphisms.

Proof. Let $\overline{G}_W = \{g \in GL(q, \mathbb{R}) : g(W) = W\}$, where g acts on $\mathfrak{so}(q, \mathbb{R})$ by $g(Z) = gZg^t$ for all $Z \in \mathfrak{so}(q, \mathbb{R})$. By Proposition 6.1.2 below the Lie group \overline{G}_W is self adjoint in either case, and in particular $\overline{G}_W = \overline{G}_{W^\perp}$ by the remarks following Proposition 3.4.5.

Let \langle, \rangle denote the canonical inner product on $\mathfrak{N} = \mathbb{R}^q \oplus W$. Let φ be any automorphism of \mathfrak{N} . By Propositions 3.4.2 and 6.2.2 φ can be written as a composition $\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3$, where φ_1, φ_2 and φ_3 are automorphisms of \mathfrak{N} such that $\varphi_1 \in U$ is unipotent, φ_2 is orthogonal relative to \langle, \rangle and φ_3 is symmetric and positive definite relative to \langle, \rangle . Since $\det(\varphi) = \pm 1$ it follows that $\det(\varphi_3) = 1$.

We note that $\psi(W) = W$ for all $\psi \in \text{Aut}(\mathfrak{N})$ since $W = [\mathfrak{N}, \mathfrak{N}]$. Let $\xi \in W^\mathbb{C}$ be a nonzero vector such that $\varphi(\xi) = \lambda\xi$ for some $\lambda \in \mathbb{C}$. We observe that $\varphi_1 \equiv \text{Id}$ on $W^\mathbb{C}$ by the definition of U and $\varphi_3 \equiv \text{Id}$ on $W^\mathbb{C}$ by (6.2.2) and (6.2.4) below. Hence $\lambda\xi = \varphi(\xi) = \varphi_2(\xi)$ and it follows that $|\lambda| = 1$ since φ_2 is orthogonal. \square

4. THE MODULI SPACE $\mathfrak{N}(p, q)$ OF 2-STEP NILPOTENT LIE ALGEBRAS OF TYPE (p, q)

We retain the notation of section 3. We explain how to identify the set $\mathfrak{N}(p, q)$ of isomorphism classes of 2-step nilpotent Lie algebras of type (p, q) with the orbit space of a natural action of $GL(q, \mathbb{R})$ on the Grassmann manifold $G(p, \mathfrak{so}(q, \mathbb{R}))$. There is a natural bijection between $\mathfrak{N}(p, q)$ and $\mathfrak{N}(D-p, q)$, where $D = (1/2)q(q-1)$.

4.1. Quotients of free 2-step nilpotent Lie algebras. Let $G(p, \mathfrak{so}(q, \mathbb{R}))$ denote the Grassmann manifold of p -dimensional subspaces of $\mathfrak{so}(q, \mathbb{R})$. The group $GL(q, \mathbb{R})$ acts on $\mathfrak{so}(q, \mathbb{R})$ by $g(Z) = gZg^t$ for all $g \in GL(q, \mathbb{R})$ and all $Z \in \mathfrak{so}(q, \mathbb{R})$. This action extends to an action of $GL(q, \mathbb{R})$ on $G(p, \mathfrak{so}(q, \mathbb{R}))$. Let $X(p, q) = G(p, \mathfrak{so}(q, \mathbb{R})) / GL(q, \mathbb{R})$ and let $P : G(p, \mathfrak{so}(q, \mathbb{R})) \rightarrow X(p, q)$ denote the projection map.

If $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$, then the standard, metric, 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$ of type (p, q) is isomorphic to $\mathfrak{F}_2(q) / W^\perp$ as we observed in (3.4). If $W_1, W_2 \in G(p, \mathfrak{so}(q, \mathbb{R}))$ are arbitrary, then $\mathfrak{N}_1 = \mathbb{R}^q \oplus W_1$ and $\mathfrak{N}_2 = \mathbb{R}^q \oplus W_2$ are isomorphic $\Leftrightarrow g(W_1^\perp) = W_2^\perp$ for some $g \in GL(q, \mathbb{R})$. Moreover, $g(W_1^\perp) = W_2^\perp \Leftrightarrow h(W_1) = W_2$, where $h = (g^t)^{-1}$. Hence we obtain

$$\begin{aligned} (*) \quad & \mathfrak{F}_2(q) / W_1^\perp \text{ is isomorphic as a Lie algebra to } \mathfrak{F}_2(q) / W_2^\perp \\ & \Leftrightarrow \mathfrak{F}_2(q) / W_1 \text{ is isomorphic as a Lie algebra to } \mathfrak{F}_2(q) / W_2 \end{aligned}$$

See Proposition 3.5b and Corollary 3.5c of [E4] for details.

If \mathfrak{N} is 2-step nilpotent of type (p, q) , then we let $[\mathfrak{N}]$ denote the isomorphism class of \mathfrak{N} . We define $\mathfrak{N}(p, q)$ to be the set of isomorphism classes of 2-step nilpotent Lie algebras of type (p, q) .

Proposition. *Let $F : X(p, q) \rightarrow \mathfrak{N}(p, q)$ be the map given by $F(P(W)) = [\mathfrak{F}_2(q) / W^\perp]$. Then F is well defined and a bijection. The spaces $X(p, q)$ and $X(D-p, q)$, $D = (1/2)q(q-1)$, are homeomorphic under the map that sends $P(W)$ to $P(W^\perp)$.*

The second statement in the result above is a consequence of (*). We call the corresponding bijection between $\mathfrak{N}(p, q)$ and $\mathfrak{N}(D-p, q)$ *duality* and the pairs (p, q) and $(D-p, q)$ *dual pairs*. For a proof of the proposition see (5.2) of [E4].

Remark. Let $\overline{G} = GL(q, \mathbb{R})$ and $G = SL(q, \mathbb{R})$. If $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$, then by Proposition 3.1.4 either $\overline{G}(W) = G(W)$ or $\overline{G}(W) = G(W) \cup G(\alpha(W))$, where α is any element of \overline{G} with $\det \alpha < 0$. In either case the codimension in $G(p, \mathfrak{so}(q, \mathbb{R}))$ of the orbits $\overline{G}(W)$ and $G(W)$ is the same. It is considerably easier to work with the action of the semisimple group $SL(q, \mathbb{R})$ than with $GL(q, \mathbb{R})$. This is the point of view of [E4], and we continue to follow that point of view in this article.

If $Y(p, q) = G(p, \mathfrak{so}(q, \mathbb{R})) / SL(q, \mathbb{R})$, then there is an obvious surjective map from $Y(p, q)$ to $X(p, q) = G(p, \mathfrak{so}(q, \mathbb{R})) / GL(q, \mathbb{R})$, and each fiber contains either one point or two.

4.2. Computing isotropy groups of $SL(q, \mathbb{R})$ acting on $G(p, \mathfrak{so}(q, \mathbb{R}))$. The space $Y(p, q) = G(p, \mathfrak{so}(q, \mathbb{R})) / SL(q, \mathbb{R})$ has a natural topology, the quotient topology. However, this topology is not Hausdorff since not all orbits of $G = SL(q, \mathbb{R})$ are closed in $G(p, \mathfrak{so}(q, \mathbb{R}))$. For example, if W' lies in the closure of an orbit $G(W)$, then any open set in $Y(p, q)$ that contains $G(W')$ also contains $G(W)$.

Following the philosophy of geometric invariant theory we would like to study the space $Y(p, q)$ by first determining the closed orbits of G in $G(p, \mathfrak{so}(q, \mathbb{R}))$. At present we have no progress to report on this problem although we have some information on a linearized version of it (see the discussion next and in section 7). In any event the first step in studying the orbit structure of G on $G(p, \mathfrak{so}(q, \mathbb{R}))$ is to determine the stabilizer in G at W of an orbit $G(W)$, where $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$. This is the goal in the remainder of section 4.

If $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$, then the orbit $G(W)$ in $G(p, \mathfrak{so}(q, \mathbb{R}))$ is bijectively equivalent to G / G_W , where $G_W = \{g \in G : g(W) = W\}$. To determine G_W it is convenient to replace the action of $G = SL(q, \mathbb{R})$ on $G(p, \mathfrak{so}(q, \mathbb{R}))$ by a linear action of $G \times H = SL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on the real vector space $V = \mathfrak{so}(q, \mathbb{R})^p = \mathfrak{so}(q, \mathbb{R}) \times \dots \times \mathfrak{so}(q, \mathbb{R})$ (p -times). Let $S = (S^1, \dots, S^p)$ be an element of V . For an element g of $G = SL(q, \mathbb{R})$ we define $g(S) = (g(S^1), \dots, g(S^p))$, where $g(Z) = gZg^t$ as usual. For an element h of $H = GL(p, \mathbb{R})$ we define $h(S) = S' = (S'^1, \dots, S'^p)$, where $S'_{\alpha\beta} = h((S^1)_{\alpha\beta}, \dots, (S^p)_{\alpha\beta})$ for $1 \leq \alpha, \beta \leq q$.

We may also identify V with $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ under the linear isomorphism that maps $S = (S^1, \dots, S^p)$ to the element $\sum_{i=1}^p S^i \otimes e_i$, where $\{e_1, \dots, e_p\}$ is the natural basis of \mathbb{R}^p . With this identification the group $G \times H$ acts on $V = \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ by $(g, h)(Z \otimes v) = g(Z) \otimes h(v)$ for all $(g, h) \in G \times H$, $Z \in \mathfrak{so}(q, \mathbb{R})$ and $v \in \mathbb{R}^p$. In this formulation it is clear that the actions of G and H commute on V .

Let $V = \mathfrak{so}(q, \mathbb{R})^p$, and let $V_o = \{S = (S^1, \dots, S^p) : \{S^1, \dots, S^p\} \text{ are linearly independent elements of } \mathfrak{so}(q, \mathbb{R})\}$. It is easy to see that V_o is invariant under $G \times H$. Moreover, V_o is a Zariski open subset of V and in particular V_o is open and dense in the Euclidean topology of V . We define a C^∞ surjective map $Q : V_o \rightarrow G(p, \mathfrak{so}(q, \mathbb{R}))$ by $Q(S) = \text{span} \{S^1, \dots, S^p\}$. Note that a fiber $Q^{-1}(W)$ consists of all bases $\{S^1, \dots, S^p\}$ of W and $H = GL(p, \mathbb{R})$ acts simply transitively on each fiber of Q . We are now ready to state a result that can be used to compute the stability group $G_W = \{g \in G = SL(q, \mathbb{R}) : g(W) = W\}$.

Proposition. *Let $G = SL(q, \mathbb{R})$ and $H = GL(p, \mathbb{R})$. Given an element W of $G(p, \mathfrak{so}(q, \mathbb{R}))$, and a basis $\{S^1, \dots, S^p\}$ of W we set $S = (S^1, \dots, S^p) \in V = \mathfrak{so}(q, \mathbb{R})^p$. Let $\pi_1 : G \times H \rightarrow G$ be projection on the first factor. Then π_1 is an isomorphism from $(G \times H)_S$ onto G_W , where $(G \times H)_S = \{(g, h) \in G \times H : (g, h)(S) = S\}$.*

Proof. Let $(g, h)(S) = S$. Since H leaves invariant the fiber $Q^{-1}(g(W))$ it follows that $\{g(S^1), \dots, g(S^p)\}$ is a basis of W . In particular $g(W) = W$, which proves that $\pi_1(G \times H)_S \subseteq G_W$. Conversely, if $g \in G_W$, then $\{g(S^1), \dots, g(S^p)\}$ is a basis of W and there exists a unique element h of $H = GL(p, \mathbb{R})$ such that $h(g(S)) = S$. This shows that $(g, h) \in (G \times H)_S$, which proves that π_1 is surjective. Since H acts simply transitively on the fibers of Q it follows that π_1 is injective. \square

4.3. Isotropy groups of $SL(q, \mathbb{R})$ of minimal dimension. Let $SL(q, \mathbb{R})$ act on $G(p, \mathfrak{so}(q, \mathbb{R}))$ as above, and let $d(W)$ be the dimension of the isotropy group $SL(q, \mathbb{R})_W$. Let $d(p, q) = \min \{d(W) : W \in G(p, \mathfrak{so}(q, \mathbb{R}))\}$. It is well known that there exists a dense open subset O of $G(p, \mathfrak{so}(q, \mathbb{R}))$ such that $d(W) = d(p, q)$ for all W in O . If $p \geq 3$, then in general $d(p, q) = 0$. In the table below we present a complete list of the pairs (p, q) for which $d(p, q)$ is positive. The integer d is always the same for "dual" pairs (p, q) and $(D-p, q)$, where $D = (1/2)q(q-1)$. See (5.4d) of [E4] for further details.

Remark. If $d(p, q) = 0$, then the isotropy group $SL(q, \mathbb{R})_W$ is finite for all W in a dense open subset of $G(p, \mathfrak{so}(q, \mathbb{R}))$ since $SL(q, \mathbb{R})_W$ is a real algebraic group.

TABLE
 $d(p,q) = 0$ except in the following cases

	(p, q)		$d(p, q)$
1)	$(1, q)$ $(D-1, q)$		$(1/2)q(q+1)$
2)	(D, q)		$q^2 - 1$ (free 2-step on q generators)
3)	$(2, 2k+1)$ $(D-2, 2k+1)$		$2k+4$
4)	$(2, 4)$ $(4, 4)$		7
5)	$(2, 2k)$ $(D-2, 2k)$	$k \geq 3$	$3k$
6)	$(3, 4)$	(self dual)	6
7)	$(3, 5)$ $(7, 5)$		3
8)	$(3, 6)$ $(12, 6)$		1

5. RATIONAL STRUCTURES IN 2-STEP NILPOTENT LIE ALGEBRAS

5.1. Rational structures in nilpotent Lie algebras. If \mathfrak{h} is a finite dimensional Lie algebra over \mathbb{R} , then \mathfrak{h} is said to have a *rational structure* if there exists a basis $\mathfrak{B} = \{\xi_1, \dots, \xi_n\}$ for \mathfrak{h} with rational structure constants; that is, $[\xi_i, \xi_j] = \sum_{k=1}^n C_{ij}^k \xi_k$, where $C_{ij}^k \in \mathbb{Q}$ for all i, j, k . Equivalently, if $\mathfrak{h}_{\mathbb{Q}} = \mathbb{Q}\text{-span}(\mathfrak{B})$, then \mathfrak{h} is a Lie algebra over \mathbb{Q} .

Let $\mathfrak{B}_1, \mathfrak{B}_2$ be two bases of \mathfrak{h} with rational structure constants. We declare the rational structures $\mathbb{Q}\text{-span}(\mathfrak{B}_1)$ and $\mathbb{Q}\text{-span}(\mathfrak{B}_2)$ to be *equivalent* if there exists an automorphism ϕ of \mathfrak{h} such that $\phi(\mathbb{Q}\text{-span}(\mathfrak{B}_1)) = \mathbb{Q}\text{-span}(\mathfrak{B}_2)$. We are interested in the problem of describing the space of equivalence classes of rational structures on a 2-step nilpotent Lie algebra \mathfrak{N} .

5.2. Scarcity of rational structures. When a simply connected nilpotent Lie group N admits a lattice Γ , then one can study dynamics on the compact quotient $\Gamma \backslash N$, or geometry if one equips N with a left invariant Riemannian metric. If the Lie algebra \mathfrak{N} arises from a Lie triple system, then we know from (1.3) that \mathfrak{N} admits a rational structure and N admits a lattice.

However, a randomly chosen group N will in general not admit a lattice Γ . This is easy to explain even for 2-step nilpotent Lie groups N . Fix positive integers p and q , and consider the space $\mathfrak{N}(p, q)$ of isomorphism classes of 2-step nilpotent Lie algebras of type (p, q) . By the discussion in (4.1) $\mathfrak{N}(p, q)$ can be identified bijectively with the orbit space $X(p, q) = G(p, \mathfrak{so}(q, \mathbb{R})) / \text{GL}(q, \mathbb{R})$. Since the rational numbers are countable the Mal'cev criterion (see (1.2)) says that there are only countably many isomorphism classes in $\mathfrak{N}(p, q)$ that admit rational structures. However, if $p \geq 3$, then it is easy to see that for q sufficiently large the codimension of any $\text{GL}(q, \mathbb{R})$ orbit in $G(p, \mathfrak{so}(q, \mathbb{R}))$ will be positive. See Proposition B in (5.4d) of [E4] for details. In particular, a random orbit will not be one of the countably many that correspond to Lie algebras \mathfrak{N} with rational structures.

Although the set of 2-step nilpotent Lie algebras of type (p, q) with a rational structure is a countable subset of $\mathfrak{N}(p, q)$ it is nevertheless dense in $\mathfrak{N}(p, q)$ as we explain in the next section.

5.3. Density of rational structures.

Definition. A p -dimensional subspace W of $\mathfrak{so}(q, \mathbb{R})$ will be called *standard rational* if W has a basis with elements in $\mathfrak{so}(q, \mathbb{Q})$.

It is easy to see that if W is a standard rational subspace of $\mathfrak{so}(q, \mathbb{R})$, then the corresponding standard, metric, 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$ admits a rational structure. If $\{Z_1, \dots, Z_p\}$ are elements of $\mathfrak{so}(q, \mathbb{Q})$ that form a basis of W , then $\mathfrak{B} = \{e_1, \dots, e_q, Z_1, \dots, Z_p\}$ is a basis of \mathfrak{N} with rational structure constants. The next result gives a converse to this observation.

Proposition 5.3.1. *Let \mathfrak{N} be a 2-step nilpotent Lie algebra of type (p, q) .*

1) *If \mathfrak{N} has a rational structure, then \mathfrak{N} is isomorphic to a standard, metric, 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$ where W is a p -dimensional standard rational subspace of $\mathfrak{so}(q, \mathbb{R})$.*

2) *Let W be a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$ and let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be the corresponding standard, metric, 2-step nilpotent Lie algebra of type (p, q) . Then \mathfrak{N} admits a rational structure $\Leftrightarrow gWg^t$ is a standard rational subspace of $\mathfrak{so}(q, \mathbb{R})$ for some $g \in GL(q, \mathbb{R})$.*

Since any subspace W of $\mathfrak{so}(q, \mathbb{R})$ is a limit in $G(p, \mathfrak{so}(q, \mathbb{R}))$ of standard rational subspaces $\{W_n\}$ we immediately obtain the following

Corollary 5.3.2. *Let \mathfrak{N} be any 2-step nilpotent Lie algebra of type (p, q) , and let $[\mathfrak{N}]$ denote its isomorphism class in $\mathfrak{N}(p, q) = X(p, q) = G(p, \mathfrak{so}(q, \mathbb{R})) / GL(q, \mathbb{R})$. In any neighborhood O of $[\mathfrak{N}]$ in $X(p, q)$ one can find a 2-step nilpotent Lie algebra \mathfrak{N}' of type (p, q) with a rational structure.*

Proof. 1) Let \mathfrak{N} admit a rational structure. We show that \mathfrak{N} admits a basis \mathfrak{B} with rational structure constants $\{C_{ij}^k\}$ that is adapted in the terminology of (3.1). The proof of Proposition 3.1.1 then shows that \mathfrak{N} is isomorphic to $\mathbb{R}^q \oplus W$, where $W = \text{span} \{C^1, \dots, C^p\}$ is a standard rational p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$.

By hypothesis \mathfrak{N} admits a basis \mathfrak{B}' such that $\mathfrak{N}_{\mathbb{Q}} = \mathbb{Q} - \text{span}(\mathfrak{B}')$ is a Lie algebra over \mathbb{Q} . Let $\mathfrak{B} = \{X_1, \dots, X_q, Z_1, \dots, Z_p\}$ be a \mathbb{Q} -basis of $\mathfrak{N}_{\mathbb{Q}}$ such that $\{Z_1, \dots, Z_p\}$ is a \mathbb{Q} -basis of $[\mathfrak{N}_{\mathbb{Q}}, \mathfrak{N}_{\mathbb{Q}}]$. Since $\dim_{\mathbb{Q}} \mathfrak{N}_{\mathbb{Q}} = \dim_{\mathbb{R}} \mathfrak{N}$ it follows that $\{Z_1, \dots, Z_p\}$ is an \mathbb{R} -basis of $[\mathfrak{N}, \mathfrak{N}]$. In particular $p = p'$, $q = q'$ and \mathfrak{B} is the desired adapted basis with rational structure constants.

2) Let W_1 and W_2 be p -dimensional subspaces of $\mathfrak{so}(q, \mathbb{R})$, and let $\mathfrak{N}_1 = \mathbb{R}^q \oplus W_1$ and $\mathfrak{N}_2 = \mathbb{R}^q \oplus W_2$ be the corresponding standard, metric, 2-step nilpotent Lie algebras of type (p, q) . By Proposition 3.1 of [E4] \mathfrak{N}_1 and \mathfrak{N}_2 are isomorphic as Lie algebras $\Leftrightarrow W_2 = gW_1g^t$ for some $g \in GL(q, \mathbb{R})$. The desired result now follows from 1). \square

5.4. Equivalence of rational structures in 2-step nilpotent Lie algebras of type (p, q) . Let \mathfrak{N} be a 2-step nilpotent Lie algebra of type (p, q) that has a rational structure. We wish to describe the space of equivalence classes of rational structures on \mathfrak{N} in a way that will make this space more computable. By the discussion in (5.3) we may assume that \mathfrak{N} is a standard, metric, 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$ where W is a p -dimensional standard rational subspace of $\mathfrak{so}(q, \mathbb{R})$.

Rational bases. Let \mathfrak{B}_o denote the standard basis $\{e_1, \dots, e_p\}$ of \mathbb{R}^q . Let $\mathfrak{B}_W = \{Z_1, \dots, Z_p\}$ be a basis of W consisting of elements in $\mathfrak{so}(q, \mathbb{Q})$. Henceforth we call such a basis B_W a *standard rational basis* of W and the basis $\mathfrak{B} = \mathfrak{B}_o \cup \mathfrak{B}_W$ a *standard rational basis* of $\mathfrak{N} = \mathbb{R}^q \oplus W$.

More generally, a basis \mathfrak{B}_q of \mathbb{R}^q will be called a *rational basis* of \mathbb{R}^q if there exists a basis $\mathfrak{B}_W = \{Z_1, \dots, Z_p\}$ of W such that $\mathfrak{B} = \mathfrak{B}_q \cup \mathfrak{B}_W$ is a basis of \mathfrak{N} with rational structure constants. Equivalently, a basis \mathfrak{B}_q of \mathbb{R}^q is a rational basis of $\mathbb{R}^q \Leftrightarrow [\mathfrak{B}_q, \mathfrak{B}_q] \subseteq \mathbb{Q} - \text{span}(\mathfrak{B}_W)$. The next observation shows that the basis \mathfrak{B}_W of W is uniquely determined by a rational basis \mathfrak{B}_q of \mathbb{R}^q , up to the action of an element of $GL(q, \mathbb{Q})$.

Proposition 5.4.1. *Fix a rational basis \mathfrak{B}_q of \mathbb{R}^q . If \mathfrak{B}_W is a basis of W and $\mathfrak{B} = \mathfrak{B}_q \cup \mathfrak{B}_W$, then \mathfrak{B} has rational structure constants $\Leftrightarrow \mathfrak{B}_W$ is a \mathbb{Q} -basis of $\mathbb{Q} - \text{span}[\mathfrak{B}_q, \mathfrak{B}_q]$.*

Proof. First, let \mathfrak{B}_W be a basis of W such that $\mathfrak{B} = \mathfrak{B}_q \cup \mathfrak{B}_W$ has rational structure constants. Then $\mathbb{Q} - \text{span} [\mathfrak{B}_q, \mathfrak{B}_q] \subseteq \mathbb{Q} - \text{span}(\mathfrak{B}_W)$, and equality holds since both $\mathbb{Q} - \text{span} [\mathfrak{B}_q, \mathfrak{B}_q]$ and $\mathbb{Q} - \text{span}(\mathfrak{B}_W)$ have the same dimension over \mathbb{Q} , namely $p = \dim_{\mathbb{R}} W$. We use the fact that $[\mathfrak{B}_q, \mathfrak{B}_q]$ is an \mathbb{R} - spanning set for W since $[\mathfrak{N}, \mathfrak{N}] = [\mathbb{R}^q, \mathbb{R}^q] = W$. It follows that \mathfrak{B}_W is a \mathbb{Q} - basis of $\mathbb{Q} - \text{span}[\mathfrak{B}_q, \mathfrak{B}_q]$. Conversely, if \mathfrak{B}_W is a \mathbb{Q} - basis of $\mathbb{Q} - \text{span} [\mathfrak{B}_q, \mathfrak{B}_q]$, then $[\mathfrak{B}_q, \mathfrak{B}_q] \subseteq \mathbb{Q} - \text{span}(\mathfrak{B}_W)$ and $\mathfrak{B} = \mathfrak{B}_q \cup \mathfrak{B}_W$ is a basis of \mathfrak{N} with rational structure constants. \square

For $g \in \text{GL}(q, \mathbb{R})$ we let \mathfrak{B}_g denote the basis $\{g(e_1), \dots, g(e_q)\}$ of \mathbb{R}^q . Clearly every basis of \mathbb{R}^q can be uniquely expressed as \mathfrak{B}_g for an element g of $\text{GL}(q, \mathbb{R})$.

Proposition 5.4.2. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ where W is a standard rational subspace of $\text{so}(q, \mathbb{R})$. For $g \in \text{GL}(q, \mathbb{R})$ the basis \mathfrak{B}_g is a rational basis of $\mathbb{R}^q \Leftrightarrow g^t W$ is a standard rational subspace of $\text{so}(q, \mathbb{R})$.*

Proof. Let $\{C^1, \dots, C^p\}$ be a standard rational basis of W . Let $\{Z_1, \dots, Z_p\}$ be the basis of W such that $\langle Z_i, C^j \rangle = -\delta_{ij}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product on $\text{so}(q, \mathbb{R})$. A routine calculation (cf. Proposition 3.2b of [E4]) shows that $[e_i, e_j] = \sum_{k=1}^p C_{ij}^k Z_k$. Let $g \in \text{GL}(q, \mathbb{R})$ be given. Then $\mathfrak{B}_g = \{v_1, \dots, v_q\}$, where $v_i = g(e_i) = \sum_{k=1}^q g_{ki} e_k$ for $1 \leq i \leq q$.

Suppose first that \mathfrak{B}_g is a rational basis for \mathbb{R}^q . Let $\{Z'_1, \dots, Z'_p\}$ be a basis of W such that $[v_i, v_j] = \sum_{k=1}^p D_{ij}^k Z_k$, where $\{D_{ij}^k\} \subseteq \mathbb{Q}$. Then $W' = \text{span} \{D^1, \dots, D^p\}$ is clearly a standard rational subspace of $\text{so}(q, \mathbb{R})$. Proposition (3.1) of [E4] shows that $W' = g^t W$.

Next, suppose that $W' = g^t W$ is a standard rational subspace of $\text{so}(q, \mathbb{R})$, and let $\{D^1, \dots, D^p\}$ be a standard rational basis of W' . Since $\{C^1, \dots, C^p\}$ is a basis of W there exists $h \in \text{GL}(p, \mathbb{R})$ such that $g^t C^r g = \sum_{k=1}^p h_{kr} D_k$. Define a basis $\{Z'_1, \dots, Z'_p\}$ for W by $Z'_k = \sum_{r=1}^p h_{kr} Z_r$ for $1 \leq k \leq p$. Assertion 3) in the proof of Proposition 3.1 of [E4] now shows that $[v_i, v_j] = \sum_{k=1}^p D_{ij}^k Z'_k$ for all i, j . Hence $[\mathfrak{B}_g, \mathfrak{B}_g] \subseteq \mathbb{Q} - \text{span} \{Z'_1, \dots, Z'_p\}$. \square

Equivalence of rational bases.

Definition. *Let $\mathfrak{B}_1, \mathfrak{B}_2$ be rational bases for $\mathfrak{N} = \mathbb{R}^q \oplus W$. Let $\mathfrak{N}_{i\mathbb{Q}} = \mathbb{Q} - \text{span}(\mathfrak{B}_i) \oplus \mathbb{Q} - \text{span}([\mathfrak{B}_i, \mathfrak{B}_i])$ for $i = 1, 2$. We say that \mathfrak{B}_1 and \mathfrak{B}_2 are equivalent if there exists an automorphism $\varphi : \mathfrak{N} \rightarrow \mathfrak{N}$ such that $\varphi(\mathfrak{N}_{1\mathbb{Q}}) = \mathfrak{N}_{2\mathbb{Q}}$.*

By the discussion above both \mathfrak{N}_1 and \mathfrak{N}_2 are Lie algebras over \mathbb{Q} .

Proposition 5.4.3. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$, where W is a standard rational subspace of \mathbb{R}^q . Let $\mathfrak{B}_1, \mathfrak{B}_2$ be rational bases of \mathfrak{N} . Then \mathfrak{B}_1 and \mathfrak{B}_2 are equivalent \Leftrightarrow there exists $g \in \text{GL}(q, \mathbb{R})$ such that*

- 1) $g(\mathbb{Q} - \text{span}(\mathfrak{B}_1)) = \mathbb{Q} - \text{span}(\mathfrak{B}_2)$
- 2) $g^t W = W$.

Proof. Suppose first that \mathfrak{B}_1 and \mathfrak{B}_2 are equivalent rational bases, and let $\varphi \in \text{Aut}(\mathfrak{N})$ be an element such that $\varphi(\mathfrak{N}_{1\mathbb{Q}}) = \mathfrak{N}_{2\mathbb{Q}}$. We prove that properties 1) and 2) hold.

Let $\mathfrak{F}_2(q) = \mathbb{R}^q \oplus \text{so}(q, \mathbb{R})$ be the free 2-step nilpotent Lie algebra discussed in (3.2), and let $\pi : \mathfrak{F}_2(q) \rightarrow \mathfrak{N}$ be the unique Lie algebra homomorphism such that $\pi(e_i) = e_i$ for $1 \leq i \leq q$. Hence π is the identity on \mathbb{R}^q . The automorphism φ of \mathfrak{N} may be lifted to an automorphism $\tilde{\varphi}$ of $\mathfrak{F}_2(q)$ such that $\pi \circ \tilde{\varphi} = \varphi \circ \pi$; see for example Proposition 2.2d of [E4] for further details. By the discussion in (3.3) there exist elements g in $\text{GL}(q, \mathbb{R})$ and S in $\text{Hom}(\mathbb{R}, \text{so}(q, \mathbb{R}))$ such that

- (a) $\tilde{\varphi}(v) = g(v) + S(v)$ for all $v \in \mathbb{R}$.
- $\tilde{\varphi}(Z) = gZg^t$ for all $Z \in \text{so}(q, \mathbb{R})$.

For $1 \leq i \leq q$ we observe that $\varphi(e_i) = (\varphi \circ \pi)(e_i) = (\pi \circ \tilde{\varphi})(e_i) = \pi(g(e_i) + S(e_i)) = g(e_i) + Z_i$, where $Z_i = \pi(S(e_i)) \in W$. We have shown

- (b) $\varphi(v) = g(v) + T(v)$ for all $v \in \mathbb{R}^q$, where $T = \pi \circ S \in \text{Hom}(\mathbb{R}^q, W)$.

If $\mathfrak{B}_1 = \{v_1, \dots, v_p\}$, then $\varphi(v_i) = g(v_i) + T(v_i) \in \varphi(\mathfrak{B}_1) \subseteq \varphi\{\mathbb{Q} - \text{span}(\mathfrak{B}_1) \oplus \mathbb{Q} - \text{span}([\mathfrak{B}_1, \mathfrak{B}_1])\} = \mathbb{Q} - \text{span}(\mathfrak{B}_2) \oplus \mathbb{Q} - \text{span}([\mathfrak{B}_2, \mathfrak{B}_2])$. This proves

(c) $g(v_i) \in \mathbb{Q} - \text{span}(\mathfrak{B}_2)$ and $T(v_i) \in \mathbb{Q} - \text{span}([\mathfrak{B}_2, \mathfrak{B}_2])$ for $1 \leq i \leq q$.

From the first assertion in (c) we obtain

(d) $g(\mathbb{Q} - \text{span}(\mathfrak{B}_1)) \subseteq \mathbb{Q} - \text{span}(\mathfrak{B}_2)$

and equality holds since $\mathbb{Q} - \text{span}(\mathfrak{B}_1)$ and $\mathbb{Q} - \text{span}(\mathfrak{B}_2)$ have the same dimension over \mathbb{Q} . This proves the first assertion of the Proposition.

We observed earlier that if $\{C^1, \dots, C^p\}$ is any basis of W and $\{Z_1, \dots, Z_p\}$ is the basis of W such that $\langle Z_i, C^j \rangle = -\delta_{ij}$, then $[e_i, e_j] = \sum_{k=1}^p C_{ij}^k Z_k$ for $1 \leq i, j \leq q$ and $1 \leq k \leq p$. It follows from Proposition 3.4 of [E4] that $\text{Ker}(\pi) = W^\perp$, the orthogonal complement of W in $\mathfrak{so}(q, \mathbb{R})$. By the definition of $\tilde{\varphi}$ it is evident that $\tilde{\varphi}$ leaves invariant $\text{Ker}(\pi) = W^\perp$. From (a) above it follows that $g(W^\perp) = W^\perp$, where $g(Z) = gZg^t$ for all Z in $\mathfrak{so}(q, \mathbb{R})$. Hence $(g^t)^{-1}(W) = W$, which proves the second assertion of the proposition.

Next, suppose that \mathfrak{B}_1 and \mathfrak{B}_2 are rational bases for \mathbb{R}^q such that conditions 1) and 2) of the proposition hold. By the discussion in (3.3) we may define an automorphism $\tilde{\varphi} : \mathfrak{F}_2(q) \rightarrow \mathfrak{F}_2(q)$ uniquely by the conditions that $\tilde{\varphi} = g$ on \mathbb{R}^q and $\tilde{\varphi}(Z) = g(Z) = gZg^t$ for all Z in $\mathfrak{so}(q, \mathbb{R})$. Condition 2) says that $g^t(W) = W$ and hence $g(W^\perp) = W^\perp = \text{Ker}(\pi)$, where $\pi : \mathfrak{F}_2(q) \rightarrow \mathfrak{N}$ is the unique Lie algebra homomorphism such that $\pi(v) = v$ for all $v \in \mathbb{R}^q$. Hence $\tilde{\varphi}$ induces an isomorphism $\varphi : \mathfrak{N} \rightarrow \mathfrak{N}$ such that $\pi \circ \tilde{\varphi} = \varphi \circ \pi$.

For $v \in \mathbb{R}^q$ we have $\varphi(v) = (\varphi \circ \pi)(v) = (\pi \circ \tilde{\varphi})(v) = \pi(g(v)) = g(v)$. This proves that $\varphi = g$ on \mathbb{R}^q . We obtain

(a) $\varphi(\mathbb{Q} - \text{span}(\mathfrak{B}_1)) = \mathbb{Q} - \text{span}(\mathfrak{B}_2)$

since $\varphi(\mathbb{Q} - \text{span}(\mathfrak{B}_1)) = \mathbb{Q} - \text{span}(\varphi(\mathfrak{B}_1)) = \mathbb{Q} - \text{span}(g(\mathfrak{B}_1)) = g(\mathbb{Q} - \text{span}(\mathfrak{B}_1)) = \mathbb{Q} - \text{span}(\mathfrak{B}_2)$ by condition 1) of the proposition.

Since $\varphi \in \text{Aut}(\mathfrak{N})$ it follows that $\varphi(\mathbb{Q} - \text{span}([\mathfrak{B}_1, \mathfrak{B}_1])) = \mathbb{Q} - \text{span}([\varphi(\mathfrak{B}_1), \varphi(\mathfrak{B}_1)]) \subseteq \mathbb{Q} - \text{span}([\mathfrak{B}_2, \mathfrak{B}_2])$ by (a), and equality holds since $\mathbb{Q} - \text{span}[\mathfrak{B}_1, \mathfrak{B}_1]$ and $\mathbb{Q} - \text{span}[\mathfrak{B}_2, \mathfrak{B}_2]$ have the same dimension over \mathbb{Q} . Hence we obtain

(b) $\varphi(\mathbb{Q} - \text{span}([\mathfrak{B}_1, \mathfrak{B}_1])) = \mathbb{Q} - \text{span}([\mathfrak{B}_2, \mathfrak{B}_2])$

Since $\mathfrak{N}_{i\mathbb{Q}} = \mathbb{Q} - \text{span}(\mathfrak{B}_i) \oplus \mathbb{Q} - \text{span}([\mathfrak{B}_i, \mathfrak{B}_i])$ for $i = 1, 2$ it follows from (a) and (b) that $\varphi(\mathfrak{N}_{1\mathbb{Q}}) = \mathfrak{N}_{2\mathbb{Q}}$. \square

Proposition 5.4.4. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$, where W is a standard rational subspace of \mathbb{R}^q . Let g_1, g_2 be elements of $GL(q, \mathbb{R})$ such that $\mathfrak{B}_{g_1} = \{g_1(e_1), \dots, g_1(e_q)\}$ and $\mathfrak{B}_{g_2} = \{g_2(e_1), \dots, g_2(e_q)\}$ are rational bases of \mathbb{R}^q . Then the following statements are equivalent.*

(1) \mathfrak{B}_{g_1} and \mathfrak{B}_{g_2} are equivalent rational bases of \mathbb{R}^q .

(2) There exist elements $h \in GL(q, \mathbb{R})$ and $\varphi \in GL(q, \mathbb{Q})$ such that $h \circ g_1 = g_2 \circ \varphi$ and $h^t W h = W$.

(3) There exists $\psi \in GL(q, \mathbb{Q})$ such that $\psi(g_2^t W g_2) = g_1^t W g_1$.

Proof. To simplify notation we write $\mathfrak{B}_1 = \mathfrak{B}_{g_1}$, $\mathfrak{B}_2 = \mathfrak{B}_{g_2}$ and $\mathfrak{B}_o = \{e_1, \dots, e_q\}$, the standard basis of \mathbb{R}^q . We also write $\mathfrak{B}_i = g_i(\mathfrak{B}_o) = \{g_i(e_1), \dots, g_i(e_q)\}$ for $i = 1, 2$.

(1) \Rightarrow (2). If \mathfrak{B}_1 and \mathfrak{B}_2 are equivalent rational bases of \mathbb{R}^q , then by Proposition 5.4.3 there exists $h \in GL(q, \mathbb{R})$ such that $h^t W h = W$ and $\mathbb{Q} - \text{span}(\mathfrak{B}_2) = h(\mathbb{Q} - \text{span}(\mathfrak{B}_1)) = \mathbb{Q} - \text{span}(h(\mathfrak{B}_1))$. If $\varphi = g_2^{-1} h g_1$, then $\varphi(\mathbb{Q} - \text{span}(\mathfrak{B}_o)) = g_2^{-1} \mathbb{Q} - \text{span}(h(\mathfrak{B}_1)) = \mathbb{Q} - \text{span}(\mathfrak{B}_o)$. Hence $\varphi \in GL(q, \mathbb{Q})$ and the assertions of 2) hold.

(2) \Rightarrow (3). If h and φ are the elements of (2), then set $\psi = \varphi^t$.

(3) \Rightarrow (2). Let ψ be the element of (3) and set $\varphi = \psi^t$ and $h = g_2 \varphi g_1^{-1}$.

(2) \Rightarrow (1). If (2) holds, then by the proof of Proposition 5.4.3 there exists $\psi \in \text{Aut}(\mathfrak{N})$ such that $\psi = h$ on \mathbb{R}^q . Since $\mathfrak{B}_1 = g_1(\mathfrak{B}_o)$ we obtain

$$(a) \psi(\mathfrak{B}_1) = (g_2 \circ \varphi)(\mathfrak{B}_o)$$

Note that $\varphi(\mathbb{Q} - \text{span}(\mathfrak{B}_o)) = \mathbb{Q} - \text{span}(\mathfrak{B}_o)$ since $\varphi \in \text{GL}(q, \mathbb{Q})$. From (a) we now obtain

$$(b) \psi(\mathbb{Q} - \text{span}(\mathfrak{B}_1)) = \mathbb{Q} - \text{span}(\psi(\mathfrak{B}_1)) = \mathbb{Q} - \text{span}(\mathfrak{B}_2)$$

Since $\psi \in \text{Aut}(\mathfrak{N})$ it follows from (b) that $\psi(\mathbb{Q} - \text{span}[\mathfrak{B}_1, \mathfrak{B}_1]) = \mathbb{Q} - \text{span}[\psi(\mathfrak{B}_1), \psi(\mathfrak{B}_1)] \subseteq (\mathbb{Q} - \text{span}[\mathfrak{B}_2, \mathfrak{B}_2])$ and equality holds since $\mathbb{Q} - \text{span}[\mathfrak{B}_1, \mathfrak{B}_1]$ and $\mathbb{Q} - \text{span}[\mathfrak{B}_2, \mathfrak{B}_2]$ have the same dimension over \mathbb{Q} . Hence $\psi(\mathfrak{N}_{1\mathbb{Q}}) = \mathfrak{N}_{2\mathbb{Q}}$, which proves (1). \square

5.5. The space of rational structures in a 2-step nilpotent Lie algebra. Let $\mathfrak{N} = \mathbb{R}^q \oplus W$, where W is a standard rational p -dimensional subspace of $\text{so}(q, \mathbb{R})$. Let \overline{G} denote $\text{GL}(q, \mathbb{R})$ and let $\overline{G}_{\mathbb{Q}}(W)$ denote the set of all standard rational p -dimensional subspaces of $\text{so}(q, \mathbb{R})$ that lie in the orbit $\overline{G}(W)$.

Let $\mathfrak{B}_{\mathbb{Q}}(\mathbb{R}^q, W)$ denote the set of bases \mathfrak{B} in \mathbb{R}^q that are rational relative to W . Let $[]$ denote the equivalence relation defined above in $\mathfrak{B}_{\mathbb{Q}}(\mathbb{R}^q, W)$, and let $[\mathfrak{B}]$ denote the equivalence class in $\mathfrak{B}_{\mathbb{Q}}(\mathbb{R}^q, W)$ of an element \mathfrak{B} . For an element $W' = g(W)$ in $\overline{G}_{\mathbb{Q}}(W)$ let $[W']$ denote its equivalence class in $\overline{G}_{\mathbb{Q}}(W) / \text{GL}(q, \mathbb{Q})$.

The next result shows that we may identify the orbit space $\overline{G}_{\mathbb{Q}}(W) / \text{GL}(q, \mathbb{Q})$ with the space of equivalence classes of rational structures on $\mathfrak{N} = \mathbb{R}^q \oplus W$, where W is any standard rational p -dimensional subspace of $\text{so}(q, \mathbb{R})$. If one has a good understanding of the structure of \overline{G}_W , the stabilizer of \overline{G} at W , then there is hope for determining the orbit space $\overline{G}_{\mathbb{Q}}(W) / \text{GL}(q, \mathbb{Q})$, possibly with methods involving Galois cohomology. In [E7] we investigate the structure of \overline{G}_W and its Lie algebra $\overline{\mathfrak{G}}_W$ in various special cases, including those cases where W is a Lie subalgebra of $\text{so}(q, \mathbb{R})$ or a Lie triple system that arises from a representation of a real negative definite Clifford algebra.

Proposition. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ where W is a standard rational p -dimensional subspace of $\text{so}(q, \mathbb{R})$. Let $T : \mathfrak{B}_{\mathbb{Q}}(\mathbb{R}^q, W) / [] \rightarrow \overline{G}_{\mathbb{Q}}(W) / \text{GL}(q, \mathbb{Q})$ be the map such that $T([\mathfrak{B}]) = [g^t(W)]$, where g is the unique element of $\overline{G} = \text{GL}(q, \mathbb{R})$ such that $\mathfrak{B} = \mathfrak{B}_g$ in the terminology above. Then T is a well defined bijection.*

Proof. T is well defined and injective by Proposition 5.4.4. The map T is surjective by 5.4.2. \square

6. STRUCTURE OF STABILIZER LIE ALGEBRAS $\overline{\mathfrak{G}}_W$

6.1. Criteria for \mathfrak{G}_W to be self adjoint. Let $\mathfrak{G}_W = \{X \in \text{End}(\mathbb{R}^q) : \text{trace}(X) = 0 \text{ and } X(W) \subseteq W\}$. This is the Lie algebra of the isotropy group G_W , where $G = \text{SL}(q, \mathbb{R})$.

The main goal of this section is the next result, whose proof may be found in the expanded version of this article on the author's website.

Proposition 6.1.1. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$. Then \mathfrak{G}_W is self adjoint if either of the following conditions holds :*

a) $W = \mathfrak{G}$, the Lie algebra of a compact connected subgroup G of $SO(q, \mathbb{R})$.

b) W is a subspace of $\text{so}(q, \mathbb{R})$ such that Z^2 is a negative multiple of the identity for each nonzero element Z of W .

Remark. Examples of a) arise when G is a compact connected Lie group and $\rho : G \rightarrow \text{End}(\mathbb{R}^q)$ is a representation of G . In this case $W = \text{d}\rho(\mathfrak{G})$ is the Lie algebra of $\rho(G)$ and W is a subalgebra of $\text{so}(q, \mathbb{R})$ if \mathbb{R}^q is endowed with a $\rho(G)$ -invariant metric.

Examples of b) arise $\Leftrightarrow W = j(\mathbb{R}^p)$, where $j : \mathcal{Cl}(p) \rightarrow \text{End}(\mathbb{R}^q)$ is a representation of the real negative definite Clifford algebra $\mathcal{Cl}(p)$ determined by \mathbb{R}^p . In this case q must be divisible by a

certain integer $d(p)$, where $d(8k) = 2^{4k}$, $d(8k+1) = 2^{4k+1}$, $d(8k+2) = d(8k+3) = 2^{4k+2}$ and $d(8k+a) = 2^{4k+3}$ if $4 \leq a \leq 7$. See the proof of Theorem 2.4 of [E6].

6.2. Consequences of the self adjointness of \overline{G}_W .

Structure of $\text{Aut}(\mathfrak{N})$ and $\text{Der}(\mathfrak{N})$. If $(\overline{G}_W)_o \subseteq \text{GL}(q, \mathbb{R})$ is self adjoint, then we can sharpen the statements of the canonical decompositions of $\text{Aut}(\mathfrak{N})_o$ and $\text{Der}(\mathfrak{N})$ in (3.4). In the next result we suppress the notation ρ and $d\rho$ used in (3.2) for the actions of $\text{GL}(q, \mathbb{R})$ and $\text{End}(\mathbb{R}^q)$ on $\text{so}(q, \mathbb{R})$.

Proposition 6.2.1. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ and suppose that $(\overline{G}_W)_o$ is self adjoint.*

a) *Let $\varphi \in \text{Aut}(\mathfrak{N})_o$. Then there exist unique elements g of $\text{GL}(q, \mathbb{R})$ and A of $\text{Hom}(\mathbb{R}^q, W)$ such that*

$$\begin{aligned} 1) \quad & g(W) \subseteq W \text{ and } \det g > 0 \text{ on } \mathbb{R}^q && \text{where } g \in \text{GL}(\text{so}(q, \mathbb{R})) \\ 2) \quad & \varphi(v) = g(v) + A(g(v)) && \text{for all } v \in \mathbb{R}^q. \\ & \varphi(Z) = gZg^t && \text{for all } Z \in W. \end{aligned}$$

Conversely, every element of $\text{GL}(\mathfrak{N})$ that has the form above lies in $\text{Aut}(\mathfrak{N})_o$.

b) *Let D be a derivation of \mathfrak{N} . Then there exist unique elements A of $\text{End}(\mathbb{R}^q)$ and B of $\text{Hom}(\mathbb{R}^q, W)$ such that*

$$\begin{aligned} 1) \quad & A(W) \subseteq W && \text{where } A \in \text{End}(\text{so}(q, \mathbb{R})) \\ 2) \quad & D(v) = A(v) + B(v) && \text{for all } v \in \mathbb{R}^q. \\ & D(Z) = AZ + ZA^t && \text{for all } Z \in W. \end{aligned}$$

Conversely, every element D of $\text{End}(\mathfrak{N})$ that has the form above is a derivation of \mathfrak{N} .

Proof. For an arbitrary subspace W of $\text{so}(q, \mathbb{R})$ an element \underline{g} of $\text{GL}(q, \mathbb{R})$ lies in $(\overline{G}_W)_o \Leftrightarrow \underline{g}^t$ lies in $(\overline{G}_{W^\perp})_o$. Hence when $(\overline{G}_W)_o$ is self adjoint it follows that $(\overline{G}_W)_o = (\overline{G}_{W^\perp})_o$. This means that in the canonical decompositions of $\text{Aut}(\mathfrak{N})$ and $\text{Der}(\mathfrak{N})$, described above in Propositions 3.4.2, 3.4.4 and their proofs, we may omit the projection homomorphism $\pi : \mathfrak{F}_2(q) \rightarrow \mathfrak{N}$. Proposition 3.4.2 shows that $\text{Aut}(\mathfrak{N})_o = U \cdot \overline{H}_o$ (semidirect product), where $\overline{H}_o = T((\overline{G}_{W^\perp})_o) = T((\overline{G}_W)_o)$. The result now follows immediately. \square

Compatibility of self adjointness with the KP decomposition. The remaining results in this section are needed for the discussion in (3.6). The first is the group analogue of the fact that if \overline{G}_W is self adjoint, then \overline{G}_W contains the symmetric and skew symmetric parts of each of its elements.

Proposition 6.2.2. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ and suppose that $(\overline{G}_W)_o$ is self adjoint. For every $g \in (\overline{G}_W)_o$ let k, p be the unique elements of $\text{GL}(q, \mathbb{R})$ such that $g = kp$, where $k \in O(n, \mathbb{R})$ and p is a positive definite symmetric element of $\text{GL}(q, \mathbb{R})$. Then $k \in (\overline{G}_W)_o$ and $p \in (\overline{G}_W)_o$.*

Proof. We shall need the following elementary observation

Lemma 6.2.3. *Let V be a real, finite dimensional inner product space, and let h be a positive definite, symmetric linear transformation on V . Then*

- 1) *h and h^2 have the same eigenvectors in V .*
- 2) *If W is a subspace of V such that $h^2(W) = W$, then $h(W) = W$.*

Proof of the lemma. Clearly any eigenvector of h is an eigenvector of h^2 . If $h^2(v) = \lambda v$ for some positive number λ , then write $v = v_1 + \dots + v_N$ where $h(v_i) = \lambda_i v_i$ for positive real numbers $\{\lambda_1, \dots, \lambda_N\}$. It follows that v is the sum of those v_i such that $\lambda_i = \sqrt{\lambda}$. This proves 1).

If W is a subspace of V invariant under h^2 , then W admits an orthonormal basis $\{w_1, \dots, w_p\}$ of eigenvectors of h^2 . By 1) this is also a basis of eigenvectors of h , which proves 2). \square

Proof of the proposition. Let g be an element of $(\overline{G}_W)_o$ and write $g = kp$ as above. Then $g^t g = p^2$, and $p^2 \in (\overline{G}_W)_o$ since $(\overline{G}_W)_o$ is self adjoint. Since p and p^2 are positive definite and symmetric on \mathbb{R}^q they are also positive definite and symmetric on $\mathfrak{so}(q, \mathbb{R})$ by the proposition in (3.2). Since $p^2(W) \subseteq W$ it follows from 2) of the lemma above that $p(W) \subseteq W$; that is, $p \in \overline{G}_W$. It remains only to show that $p \in (\overline{G}_W)_o$, for then $k = gp^{-1} \in (\overline{G}_W)_o$.

Since $g \in (\overline{G}_W)_o$ there exists a continuous curve $g(t)$ in $(\overline{G}_W)_o$ such that $g(0) = \text{Id}$ and $g(1) = g$. Let $g(t) = k(t)p(t)$ be the corresponding KP decomposition of $g(t)$. The argument above shows that $p(t) \in (\overline{G}_W)_o$ for all t . Since $p(0) = \text{Id}$ and $p(1) = p$ it follows that $p \in (\overline{G}_W)_o$. \square

The structure of symmetric elements of $(\overline{G}_W)_o$. The structure of the symmetric and skew symmetric elements of \overline{G}_W is studied in much greater detail in [E7]. Here we present only a result that is needed for Proposition 3.5.2.

Proposition 6.2.4. *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$, where $(\overline{G}_W)_o$ is self adjoint. Suppose that every nonzero element of W is invertible. Let p be a symmetric positive definite element of $(\overline{G}_W)_o$ such that $\det(p) = 1$. Then*

1) p is the identity on W , where $p(Z) = pZp^t = pZp$ for all Z in W .

2) If W acts irreducibly on \mathbb{R}^q , then there exists a positive real number λ such that $\mathbb{R}^q = V_\lambda \oplus V_{1/\lambda}$, where V_λ and $V_{1/\lambda}$ denote the λ and $1/\lambda$ eigenspaces of p in \mathbb{R}^q .

Proof. 1) By the proposition in (3.2) the action of p on W is positive definite and symmetric relative to the canonical inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{so}(q, \mathbb{R})$. Since $p(W) \subseteq W$ it follows that there exists an orthonormal basis $\{Z_1, \dots, Z_p\}$ of W and positive real numbers $\{\lambda_1, \dots, \lambda_p\}$ such that $p(Z_i) = \lambda_i Z_i$ for each i . It suffices to prove that $\lambda_i = 1$ for each i .

We are given that $\lambda_i Z_i = p(Z_i) = pZ_i p^t = pZ_i p$ for each i . Since Z_i is invertible it follows that $\lambda_i \text{Id} = Z_i^{-1} p Z_i p$ and taking determinants yields $\lambda_i^q = (\det p)^2 = 1$. Hence $\lambda_i = 1$ for each i since λ_i is positive.

2) Let λ be an eigenvalue for p in \mathbb{R}^q , and let v be a nonzero vector in \mathbb{R}^q such that $p(v) = \lambda v$. If Z is any element of W , then by 1) we know that $p(Z(v)) = Zp^{-1}(v) = Z((1/\lambda)v) = (1/\lambda)Z(v)$. Hence $Z(V_\lambda) \subseteq V_{1/\lambda}$ and $Z(V_{1/\lambda}) \subseteq V_\lambda$ for all Z in W . The subspace $V_\lambda \oplus V_{1/\lambda}$ is invariant under W and therefore must be equal to \mathbb{R}^q since W acts irreducibly on \mathbb{R}^q . \square

7. MINIMAL VECTORS, CLOSED ORBITS AND THE RICCI TENSOR

7.1. Minimal vectors and closed orbits of reductive groups. A Lie group G is *reductive* if the Lie algebra \mathfrak{G} is the direct sum of the center \mathfrak{Z} and a semisimple subalgebra \mathfrak{G}' . Let $G \subseteq \text{GL}(q, \mathbb{R})$ be a reductive, real algebraic group. It follows from the main result of [Mo] that there exists an element g of $\text{GL}(q, \mathbb{R})$ such that gGg^{-1} is self adjoint with respect to $\langle \cdot, \cdot \rangle$. Conversely, any self adjoint subgroup G of $\text{GL}(q, \mathbb{R})$ is easily seen to be reductive.

Let $\{V, \langle \cdot, \cdot \rangle\}$ be a real, finite dimensional, inner product space. Let $G \subseteq \text{GL}(V)$ be a connected subgroup that is self adjoint relative to $\langle \cdot, \cdot \rangle$, and let \mathfrak{G} denote the Lie algebra of G . Write $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$, where \mathfrak{K} is the subalgebra of skew symmetric elements of \mathfrak{G} and \mathfrak{P} is the subspace consisting of the symmetric elements of \mathfrak{G} . Let K be the connected subgroup of $\text{SO}(V, \langle \cdot, \cdot \rangle)$ whose Lie algebra is \mathfrak{K} . The group K is a maximal compact subgroup of G , and any other maximal compact subgroup of G is conjugate to K in G .

A vector v of V is said to be *minimal* for the group G if $|g(v)| \geq |v|$ for all $g \in G$. Clearly the zero vector is always minimal. For each nonzero vector v define a smooth function $\rho_v : G \rightarrow \mathbb{R}$ by $\rho_v(g) = |g(v)|^2$. Let \mathfrak{M} denote the set of minimal vectors in V . Clearly \mathfrak{M} is invariant under K . Note also that \mathfrak{M} is invariant under multiplication by real numbers.

There is an important relationship between the orbits $G(v)$ that are closed in V and the vectors v that are minimal in V . The proof of the following result may be found in [RS]. The topology referred to below and in [RS] is the classical topology of V and not the Zariski topology. If $A \subseteq V$ is any subset, then \overline{A} denotes the closure of A in V .

Theorem. *Let $\{V, \langle, \rangle\}$ and G be as above. Let v be a nonzero vector of V . Then*

- 1) *The vector v is minimal $\Leftrightarrow \rho_v : G \rightarrow \mathbb{R}$ has a critical point at the identity of G .*
- 2) *If $v \in \mathfrak{M}$, then $G(v) \cap \mathfrak{M} = K(v)$.*
- 3) *The orbit $G(v)$ is closed in $V \Leftrightarrow G(v) \cap \mathfrak{M}$ is nonempty.*
- 4) *For any nonzero vector v there exists a nonzero element A of \mathfrak{P} such that $v_0 = \lim_{t \rightarrow \infty} e^{tA}(v) \in \overline{G(v)} \cap \mathfrak{M}$. Hence $K(v_0) = \overline{G(v)} \cap \mathfrak{M}$.*

Remark. The vector v_0 in 4) may be the zero vector, and determining when this is the case is typically an important problem.

See [La3] as well as [RS] for applications of the result above. In this article we will use the result to study the Ricci tensor of a standard, metric, 2-step nilpotent Lie algebra $\mathfrak{N} = \mathbb{R}^q \oplus W$ of type (p, q) .

Fix positive integers $q \geq 2$ and $p \geq 2$ and let $V = \mathfrak{so}(q, \mathbb{R})^p = \mathfrak{so}(q, \mathbb{R}) \times \dots \times \mathfrak{so}(q, \mathbb{R})$ (p times). Let \langle, \rangle denote the standard inner product on $\mathfrak{so}(q, \mathbb{R})$, and let \langle, \rangle also denote the extension to V given by $\langle (C^1, \dots, C^p), (D^1, \dots, D^p) \rangle = \sum_{i=1}^p \langle C^i, D^i \rangle$. As in (4.2) we identify V with $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ under the linear isomorphism that maps $S = (S^1, \dots, S^p)$ to the element $\sum_{i=1}^p S^i \otimes e_i$, where $\{e_1, \dots, e_p\}$ is the natural basis of \mathbb{R}^p .

Recall that with this identification the group $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ becomes a subgroup of $GL(V)$ if one defines $(g, h)(Z \otimes v) = g(Z) \otimes h(v)$, where $Z \in \mathfrak{so}(q, \mathbb{R})$, $v \in \mathbb{R}^p$, $h(v)$ has the obvious definition and $g(Z) = gZg^t$. As in (4.2) we note that the actions of $GL(q, \mathbb{R})$ and $GL(p, \mathbb{R})$ commute on V . The Lie algebra $gl(q, \mathbb{R}) \times gl(p, \mathbb{R})$ becomes a subalgebra of $\text{End}(V)$ if one defines $(X, Y)(Z \otimes v) = X(Z) \otimes Y(v)$, where $Z \in \mathfrak{so}(q, \mathbb{R})$, $v \in \mathbb{R}^p$, $Y(v)$ has the obvious definition and $X(Z) = XZ + ZX^t$. Similarly, the actions of $gl(q, \mathbb{R})$ and $gl(p, \mathbb{R})$ commute on V .

Note that the metric transpose operations in $\text{End}(\mathbb{R}^q)$ and $\text{End}(V)$ are compatible with these actions of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ and $gl(q, \mathbb{R}) \times gl(p, \mathbb{R})$ on V ; that is, $\langle (g, h)(C), D \rangle = \langle C, (g^t, h^t)(D) \rangle$ for C, D in V and (g, h) in $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ or $gl(q, \mathbb{R}) \times gl(p, \mathbb{R})$. We omit the proofs of these assertions. In particular \mathfrak{K} and \mathfrak{P} act on V by skew symmetric and symmetric linear transformations respectively.

7.2. Closed orbits of $SL(q, \mathbb{R})$ and $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ on $V = \mathfrak{so}(q, \mathbb{R})^p$. By the theorem in (7.1) the group $\overline{G} = GL(q, \mathbb{R})$ has no closed orbits other than $\{0\}$ in V since \overline{G} has no nonzero minimal vectors in V ; if v is any nonzero vector in V and $g(t) = (1+t)\text{Id}$, then for $t > -1$ it is clear that $g(t) \in \overline{G}$ and $|g(t)v| = (1+t)|v|$. Similarly, $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ or $GL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ has no closed orbits in V other than $\{0\}$.

However, for the smaller groups $G = SL(q, \mathbb{R})$ and $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ the situation is quite different, as we show in the next three results. We relate these results to the Ricci tensor in (7.3) - (7.6).

Proposition 7.2.1. *Let $C = (C^1, \dots, C^p)$ be an element of $V = \mathfrak{so}(q, \mathbb{R})^p$. Then C is minimal for the action of $G = SL(q, \mathbb{R})$ on $V \Leftrightarrow \sum_{i=1}^p (C^i)^2$ is a negative multiple of the identity.*

Proposition 7.2.2. *Let $C = (C^1, \dots, C^p)$ be an element of $V = \mathfrak{so}(q, \mathbb{R})^p$. Then C is minimal for the action of $H = SL(p, \mathbb{R})$ on $V \Leftrightarrow$ there exists a positive constant c such that $\langle C^i, C^j \rangle = c \delta_{ij}$ for $1 \leq i, j \leq p$.*

As an immediate consequence of the preceding two results we obtain

Proposition 7.2.3. *Let $C = (C^1, \dots, C^p)$ be an element of $V = \mathfrak{so}(q, \mathbb{R})^p$. Then C is minimal for the action of $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ on $V \Leftrightarrow$*

- a) $\sum_{i=1}^p (C^i)^2$ is a negative multiple of the identity.
- b) There exists a positive constant c such that $\langle C^i, C^j \rangle = c \delta_{ij}$ for $1 \leq i, j \leq p$.

Remark. The third result has been proved in [La1] by a different method as a special case of a more general result.

Proof of Proposition 7.2.1 Let $\mathfrak{G} = \mathfrak{sl}(q, \mathbb{R})$ denote the Lie algebra of G , and let $X \in \mathfrak{G}$ act on $C = (C^1, \dots, C^p) \in V$ by $X(C) = (X(C^1), \dots, X(C^p))$, where $X(S) = XS + SX^t$ for $S \in \mathfrak{so}(q, \mathbb{R})$. It follows that $C = (C^1, \dots, C^p)$ is minimal for $G \Leftrightarrow \langle X(C), C \rangle = 0$ for all $X \in \mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$, where $\mathfrak{K} = \mathfrak{so}(q, \mathbb{R})$ and \mathfrak{P} is the set of symmetric elements of $\mathfrak{sl}(q, \mathbb{R})$. This condition is automatically satisfied if $X \in \mathfrak{K}$ since the elements of \mathfrak{K} act skew symmetrically on V . Hence we conclude

$$(*) C = (C^1, \dots, C^p) \text{ is minimal for } G \Leftrightarrow \langle X(C), C \rangle = 0 \text{ for all } X \in \mathfrak{P}.$$

Lemma. *Let $X \in \mathfrak{P}$ and $C = (C^1, \dots, C^p) \in V$. Let $\bar{C} = \sum_{i=1}^p (C^i)^2$. Then*

$$1) \langle X(C), C \rangle = -2 \text{ trace}(X\bar{C})$$

2) If $D = \alpha^{-1}C\alpha = (\alpha^{-1}C^1\alpha, \dots, \alpha^{-1}C^p\alpha)$ and $Y = \alpha X \alpha^{-1}$ for some $\alpha \in O(q, \mathbb{R})$, then $\langle X(D), D \rangle = \langle Y(C), C \rangle$.

Proof of the lemma The first assertion follows from the fact that $\langle X(C), C \rangle = \sum_{i=1}^p \langle X(C^i), C^i \rangle$ and $\langle X(C^i), C^i \rangle = \langle XC^i + C^iX, C^i \rangle = -2 \text{ trace}(X(C^i)^2)$. The second assertion follows immediately from the first and the fact that $\bar{D} = \alpha^{-1}\bar{C}\alpha$. \square

We now complete the proof of the proposition. First, let $C = (C^1, \dots, C^p)$ be a nonzero minimal vector for G in V . We wish to show that $\bar{C} = \sum_{i=1}^p (C^i)^2$ is a negative multiple of the identity. Note that \bar{C} is symmetric and negative semidefinite since each C^i is skew symmetric. Moreover, \bar{C} is nonzero since C is nonzero. It suffices to show that any two eigenvalues of \bar{C} are equal.

Let $\{v_1, v_2\}$ be orthogonal unit vectors in \mathbb{R}^q such that $\bar{C}v_1 = \lambda_1 v_1$ and $\bar{C}v_2 = \lambda_2 v_2$ for some real numbers λ_1, λ_2 . We show that $\lambda_1 = \lambda_2$. Choose $\alpha \in O(q, \mathbb{R})$ so that $\alpha(e_1) = v_1$ and $\alpha(e_2) = v_2$, where $\{e_1, e_2\}$ are the first two elements of the standard orthonormal basis $\{e_1, \dots, e_q\}$ of \mathbb{R}^q . Let $D = \alpha^{-1}C\alpha = (\alpha^{-1}C^1\alpha, \dots, \alpha^{-1}C^p\alpha) \in V$. Hence $D = (D^1, \dots, D^p)$, where $D^i = \alpha^{-1}C^i\alpha$ for $1 \leq i \leq p$. Let $X = \text{diag}(1, -1, 0, \dots, 0) \in \mathfrak{P}$ and let $Y = \alpha X \alpha^{-1} \in \mathfrak{P}$.

Using the lemma and the minimality of C we obtain $0 = \langle Y(C), C \rangle = \langle X(D), D \rangle = -2 \text{ trace}(X\bar{D})$, where $\bar{D} = \sum_{i=1}^p (D^i)^2 = \alpha^{-1}\bar{C}\alpha$. From the definition of X and \bar{D} it follows immediately that $(X\bar{D})(e_1) = \lambda_1 e_1$, $(X\bar{D})(e_2) = -\lambda_2 e_2$ and $(X\bar{D})(e_k) = 0$ for $k \geq 3$. Hence $0 = -2 \text{ trace}(X\bar{D}) = -2(\lambda_1 - \lambda_2)$. We have proved that \bar{C} is a negative multiple of the identity if C is minimal for $G = SL(q, \mathbb{R})$.

Conversely, suppose that \bar{C} is a negative multiple of the identity for some nonzero element C of V . Then for $X \in \mathfrak{P}$ we have $\langle X(C), C \rangle = -2 \text{ trace}(X\bar{C}) = (\text{const}) \text{ trace}(X) = 0$, which proves that C is minimal for G . \square

Proof of Proposition 7.2.2 Let $\mathfrak{H} = \mathfrak{sl}(p, \mathbb{R})$ denote the Lie algebra of $H = SL(p, \mathbb{R})$. Let $C = (C^1, \dots, C^p) \in V$. If we identify V with $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$, then $C = \sum_{i=1}^p C^i \otimes e_i$ and H acts on V by $h(C) = \sum_{i=1}^p C^i \otimes h(e_i)$. In particular, $h(S \otimes v) = S \otimes h(v)$ for $S \in \mathfrak{so}(q, \mathbb{R})$ and $v \in \mathbb{R}^p$. Hence \mathfrak{H} acts on V by $X(S \otimes v) = S \otimes X(v)$ for $X \in \mathfrak{H}, S \in \mathfrak{so}(q, \mathbb{R})$ and $v \in \mathbb{R}^p$.

If $C = \sum_{i=1}^p C^i \otimes e_i$ and $D = \sum_{j=1}^p D^j \otimes e_j$ are elements of V , then from the definition of the inner product on V we have $\sum_{i=1}^p \langle C^i, D^i \rangle = \langle C, D \rangle = \sum_{i,j=1}^p \langle C^i \otimes e_i, D^j \otimes e_j \rangle$. By choosing C, D appropriately it follows that $\langle S \otimes e_i, T \otimes e_j \rangle = \langle S, T \rangle \delta_{ij}$ for all $S, T \in \mathfrak{so}(q, \mathbb{R})$ and all i, j with $1 \leq i, j \leq p$. Hence $\langle S \otimes v, T \otimes w \rangle = \langle S, T \rangle \langle v, w \rangle$ for all $S, T \in \mathfrak{so}(q, \mathbb{R})$ and all $v, w \in \mathbb{R}^p$.

We write $\mathfrak{H} = \mathfrak{sl}(p, \mathbb{R}) = \mathfrak{K}' \otimes \mathfrak{P}'$, where $\mathfrak{K}' = \mathfrak{so}(p, \mathbb{R})$ and $\mathfrak{P}' = \{X \in \mathfrak{H} : X \text{ is symmetric}\}$. As in the proof of Proposition 7.2.1 we know that $C = \sum_{i=1}^p C^i \otimes e_i$ is minimal for $H = \text{SL}(p, \mathbb{R}) \Leftrightarrow$

$$(*) \ 0 = \langle X(C), C \rangle = \sum_{i,j=1}^p \langle C^i, C^j \rangle \langle X(e_i), e_j \rangle \text{ for all } X \in \mathfrak{P}'.$$

First we suppose that C is a minimal vector for H . Fix $i \neq j$ with $1 \leq i, j \leq p$. Let $X \in \mathfrak{P}'$ be that element such that $X(e_i) = e_j, X(e_j) = e_i$ and $X(e_k) = 0$ for $k \neq i$ and $k \neq j$. Then $(*)$ becomes

$$1) \ 0 = 2\langle C^i, C^j \rangle \text{ for all } i \neq j \text{ with } 1 \leq i, j \leq p.$$

Now fix i with $1 \leq i \leq p$, and let $X \in \mathfrak{P}'$ be that element such that $X(e_i) = e_i, X(e_{i+1}) = -e_{i+1}$ and $X(e_k) = 0$ for $k \neq i$ and $k \neq i+1$. Then $(*)$ becomes

$$2) \ 0 = \langle C^i, C^i \rangle - \langle C^{i+1}, C^{i+1} \rangle \text{ for all } i \text{ with } 1 \leq i \leq p-1.$$

It follows that there exists a real number c such that $\langle C^i, C^i \rangle = c$ for all i with $1 \leq i \leq p$. The constant c must be positive since $C^i \neq 0$ for some i . If $C = (C^1, \dots, C^p)$ is minimal, then from 1) and 2) we obtain

$$(**) \ \langle C^i, C^j \rangle = c \delta_{ij} \text{ for some positive constant } c$$

Conversely, if $(**)$ holds for $C = (C^1, \dots, C^p)$, then 1) and 2) above are satisfied. We conclude from the discussion above that $\langle X(C), C \rangle = 0$ if $X \in \mathfrak{P}'$ is either of the two types considered in 1) and 2). However, elements $X \in \mathfrak{P}'$ of these two types span \mathfrak{P}' , and it follows that $\langle X(C), C \rangle = 0$ for all $X \in \mathfrak{P}'$. By $(*)$ we conclude that $C = (C^1, \dots, C^p)$ is minimal. \square

7.3. Minimal $\text{SL}(q, \mathbb{R})$ vectors and geodesic flow invariant Ricci tensors. Let $\{\mathfrak{N}, \langle, \rangle\}$ be a metric 2-step nilpotent Lie algebra, and let $\{N, \langle, \rangle\}$ denote the simply connected, 2-step nilpotent Lie group with the corresponding left invariant metric \langle, \rangle . For all $v \in \text{TN}$ let $\text{Ric}(v) = \text{Ric}(v, v)$, where $\text{Ric}(\cdot, \cdot)$ denotes the Ricci tensor. Let $\{g^t\}$ denote the geodesic flow in TN .

We recall from (1.5) that $\{\mathfrak{N}, \langle, \rangle\}$ has a *geodesic flow invariant Ricci tensor* if $\text{Ric}(g^t v) = \text{Ric}(v)$ for all $t \in \mathbb{R}$ and all $v \in \text{TN}$. In other words $\{\mathfrak{N}, \langle, \rangle\}$ has a geodesic flow invariant Ricci tensor $\Leftrightarrow \text{Ric} : \text{TN} \rightarrow \mathbb{R}$ is a first integral for the geodesic flow.

For convenience we restate Proposition 2.5.8, which is important for the discussion here. We use the notation from (1.4).

Proposition 7.3.1. *Let $\{\mathfrak{N}, \langle, \rangle\}$ be a metric 2-step nilpotent Lie algebra, and write $\mathfrak{N} = V \oplus \mathfrak{Z}$, where \mathfrak{Z} is the center of \mathfrak{N} and $V = \mathfrak{Z}^\perp$. Suppose that $j(\mathfrak{Z})$ acts irreducibly on V . Then the Ricci tensor on $\{\mathfrak{N}, \langle, \rangle\}$ is geodesic flow invariant $\Leftrightarrow \sum_{i=1}^p j(Z_i)^2$ is a negative multiple of the identity for any orthonormal basis $\{Z_1, \dots, Z_p\}$ of \mathfrak{Z} .*

Standard 2-step examples with modified inner products on W . We now consider standard, metric, 2-step nilpotent Lie algebras $\mathfrak{N} = \mathbb{R}^q \oplus W$ of type (p, q) equipped as usual with the standard inner product on \mathbb{R}^q and the canonical inner product \langle, \rangle on $\mathfrak{so}(q, \mathbb{R})$ given by $\langle Z, Z' \rangle = -\text{trace}(ZZ')$. In what follows we sometimes need to replace the inner product \langle, \rangle on W by an arbitrary inner product \langle, \rangle^* on W . We then extend this inner product to an inner product \langle, \rangle^* on $\mathfrak{N} = \mathbb{R}^q \oplus W$ such that \langle, \rangle^* is the standard inner product on \mathbb{R}^q and the factors \mathbb{R}^q and W are orthogonal. We call this the extension of \langle, \rangle^* on W to \mathfrak{N} . As in (1.4) we then define a linear map $j^* : \mathfrak{Z} \rightarrow \mathfrak{so}(\mathbb{R}^q, \langle, \rangle^*)$ by $\langle j^*(Z) X, Y \rangle^* = \langle [X, Y], Z \rangle^*$ for $Z \in W$ and $X, Y \in \mathbb{R}^q$

Proposition 7.3.2. *Let W be a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$. Then the following statements are equivalent.*

- 1) W admits a basis $\{C^1, \dots, C^p\}$ such that

$$\sum_{i=1}^p (C^i)^2 = -\lambda^2 \text{Id} \quad \text{for some positive number } \lambda.$$
- 2) Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be the standard, metric, 2-step nilpotent Lie algebra determined by W . Then
 - a) W is the center of \mathfrak{N} .

b) W admits an inner product $\langle \cdot, \cdot \rangle^*$ such that $\{\mathfrak{N}, \langle \cdot, \cdot \rangle^*\}$ has geodesic flow invariant Ricci tensor, where $\langle \cdot, \cdot \rangle^*$ denotes the extension of $\langle \cdot, \cdot \rangle$ to \mathfrak{N} .

3) W admits a basis $\{C^1, \dots, C^p\}$ such that $C = (C^1, \dots, C^p)$ is a minimal vector for $SL(q, \mathbb{R})$ acting on $V = \mathfrak{so}(q, \mathbb{R})^p = \mathfrak{so}(q, \mathbb{R}) \times \dots \times \mathfrak{so}(q, \mathbb{R})$ (p times).

Proof. The equivalence of 1) and 3) is Proposition 7.2.1. Hence we need only prove the equivalence of 1) and 2).

1) \Rightarrow 2). Clearly $W = [\mathfrak{N}, \mathfrak{N}]$ lies in the center of \mathfrak{N} , and it is straightforward to show that equality holds $\Leftrightarrow W$ has trivial kernel in \mathbb{R}^q . Suppose that 1) holds and $Z(v) = 0$ for all Z in W and some v in \mathbb{R}^q . Then $-\lambda^2 v = \sum_{i=1}^p (C^i)^2(v) = 0$ for some positive number λ . Hence $v = 0$, which proves a).

We prove b). Let $\langle \cdot, \cdot \rangle$ denote the canonical inner product on $\mathfrak{so}(q, \mathbb{R})$. Let $\{C^1, \dots, C^p\}$ be a basis for W as in 1), and let $\{\rho_1, \dots, \rho_p\}$ be the basis in W such that $\langle C^i, \rho_j \rangle = -\delta_{ij}$. By Proposition 3.2b in [E4] we have

$$(i) [e_i, e_j] = \sum_{k=1}^p C_{ij}^k \rho_k$$

where $\{e_1, \dots, e_q\}$ is the standard orthonormal basis of \mathbb{R}^q . Now let $\langle \cdot, \cdot \rangle^*$ be the inner product on W that makes $\{\rho_1, \dots, \rho_p\}$ an orthonormal basis of W relative to $\langle \cdot, \cdot \rangle^*$, and let $\langle \cdot, \cdot \rangle^*$ also denote the extension of $\langle \cdot, \cdot \rangle^*$ to $\mathfrak{N} = \mathbb{R}^q \oplus W$. If $j^* : W = \mathfrak{Z} \rightarrow \mathfrak{so}(\mathbb{R}^q, \langle \cdot, \cdot \rangle^*)$ is the linear map defined by $\langle j^*(Z) X, Y \rangle^* = \langle [X, Y], Z \rangle^*$ for all $Z \in W$ and $X, Y \in \mathbb{R}^q$, then from the definitions and (i) we obtain

$$(ii) j^*(\rho_k) \text{ has matrix } -C^k \text{ relative to } \{e_1, \dots, e_q\} \text{ for } 1 \leq k \leq p.$$

Finally, from (ii) and the hypothesis of 1) it follows that $\sum_{k=1}^p j^*(\rho_k)^2$ is a negative multiple of the identity. From Proposition 7.3.1 it follows that $\{\mathfrak{N}, \langle \cdot, \cdot \rangle^*\}$ has geodesic flow invariant Ricci tensor, which completes the proof of 1) \Rightarrow 2).

We prove 2) \Rightarrow 1), essentially by reversing the argument of 1) \Rightarrow 2). Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be a standard, metric, 2-step nilpotent Lie algebra satisfying a) and b) of 2). By a) we know that $W = \mathfrak{Z}$, the center of \mathfrak{N} . Let $\langle \cdot, \cdot \rangle^*$ be the metric on W with the properties of 2b), and let $\{\rho_1, \dots, \rho_p\}$ be a basis of W that is orthonormal with respect to $\langle \cdot, \cdot \rangle^*$. Let $\{C^1, \dots, C^p\}$ be a basis for W such that $\langle C^i, \rho_j \rangle = -\delta_{ij}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product in $\mathfrak{so}(q, \mathbb{R})$. By the choice of $\{C^1, \dots, C^p\}$ it follows as in (i) above that

$$[e_i, e_j] = \sum_{k=1}^p C_{ij}^k \rho_k$$

If $j^* : \mathfrak{Z} \rightarrow \mathfrak{so}(\mathbb{R}^q, \langle \cdot, \cdot \rangle^*)$ is defined as above, then it is straightforward to check that $j^*(\rho_k)$ has matrix $-C^k$ relative to $\{e_1, \dots, e_q\}$ for $1 \leq k \leq p$. By the hypothesis 2b) and Proposition 7.3.1 we know that $\sum_{k=1}^p j^*(\rho_k)^2 = -\lambda^2 \text{Id}$ for some positive number λ . Hence $\sum_{k=1}^p (C^k)^2 = -\lambda^2 \text{Id}$. \square

7.4. Closed $SL(q, \mathbb{R})$ orbits and geodesic flow invariant Ricci tensor.

Proposition 7.4.1. *Let $SL(q, \mathbb{R})$ act on $V = \mathfrak{so}(q, \mathbb{R})^p = \mathfrak{so}(q, \mathbb{R}) \times \dots \times \mathfrak{so}(q, \mathbb{R})$ (p times) as defined at the end of (7.1). Let W be a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$. Then the following statements are equivalent.*

- 1) For some basis $\{C^1, \dots, C^p\}$ of W the orbit $SL(q, \mathbb{R})(C)$ is closed in V , where $C = (C^1, \dots, C^p)$.
- 2) For every basis $\{C^1, \dots, C^p\}$ of W the orbit $SL(q, \mathbb{R})(C)$ is closed in V , where $C = (C^1, \dots, C^p)$.
- 3) Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be the standard, metric, 2-step nilpotent Lie algebra determined by W .

Then

- a) W is the center of \mathfrak{N} .
- b) \mathfrak{N} admits an inner product $\langle \cdot, \cdot \rangle^*$ such that $\{\mathfrak{N}, \langle \cdot, \cdot \rangle^*\}$ has geodesic flow invariant Ricci tensor.

4) There exists $g \in SL(q, \mathbb{R})$ such that if $W' = g(W) = gWg^t$, then W' admits a basis $\{D^1, \dots, D^p\}$ such that $\sum_{i=1}^p (D^i)^2 = -\lambda^2 \text{Id}$ for some positive number λ .

Proof. We prove this result by showing 4) \Rightarrow 3), 3) \Rightarrow 1), 1) \Rightarrow 2) and 2) \Rightarrow 4).

4) \Rightarrow 3). Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ and $\mathfrak{N}' = \mathbb{R}^q \oplus W'$ be the standard, metric, 2-step nilpotent Lie algebras defined by the subspaces W and W' . By Proposition 7.3.2 \mathfrak{N}' admits an inner product $\langle \cdot, \cdot \rangle'$ such that $\{\mathfrak{N}', \langle \cdot, \cdot \rangle'\}$ has geodesic flow invariant Ricci tensor. By Proposition 3.1.3 there exists a Lie algebra isomorphism $T : \mathfrak{N} \rightarrow \mathfrak{N}'$. Let $\langle \cdot, \cdot \rangle^*$ be the inner product on \mathfrak{N} that makes T an isometry. Then $\{\mathfrak{N}, \langle \cdot, \cdot \rangle^*\}$ has geodesic flow invariant Ricci tensor, which proves b) in 3).

It remains only to prove a). By the proof of Proposition 7.3.2 the subspace W' has trivial kernel in \mathbb{R}^q , which proves that W' is the center of \mathfrak{N}' . In particular, \mathfrak{N}' has p -dimensional center, and the same is true for the isomorphic Lie algebra \mathfrak{N} . Since $W = [\mathfrak{N}, \mathfrak{N}]$ is a p -dimensional subspace of the center of \mathfrak{N} it follows that W is the center of \mathfrak{N} .

3) \Rightarrow 1). Suppose first that $\mathfrak{N} = \mathbb{R}^q \oplus W$ admits some metric $\langle \cdot, \cdot \rangle^*$ whose Ricci tensor is geodesic flow invariant. By Proposition 3.1.2 there exists a standard, metric, 2-step nilpotent Lie algebra $\mathfrak{N}' = \mathbb{R}^q \oplus W'$ and an additional inner product $\langle \cdot, \cdot \rangle'$ on \mathfrak{N}' such that $\langle \cdot, \cdot \rangle'$ is the standard inner product on \mathbb{R}^q , the subspaces W and \mathbb{R}^q are orthogonal relative to $\langle \cdot, \cdot \rangle'$ and $\{\mathfrak{N}', \langle \cdot, \cdot \rangle^*\}$ is both isometric and Lie algebra isomorphic to $\{\mathfrak{N}', \langle \cdot, \cdot \rangle'\}$. It follows from Proposition 3.1.3 that $W' = g(W) = gWg^t$ for some $g \in GL(q, \mathbb{R})$ since $\mathfrak{N} = \mathbb{R}^q \oplus W$ and $\mathfrak{N}' = \mathbb{R}^q \oplus W'$ are isomorphic.

By the implication 2) \Rightarrow 1) in Proposition 7.3.2 it follows that W' has a basis $\{E^1, \dots, E^p\}$ such that $\sum_{i=1}^p (E^i)^2 = -\lambda^2 \text{Id}$ for some positive number λ . Hence $E = (E^1, \dots, E^p)$ is a minimal vector for $SL(q, \mathbb{R})$ in $V = \mathfrak{so}(q, \mathbb{R})^p$ by Proposition 7.2.1. It follows from 3) of the theorem in (7.1) that the orbit $SL(q, \mathbb{R})(E)$ is closed in V . If $C^i = g^{-1}(E^i)$ for $1 \leq i \leq p$, then $\{C^1, \dots, C^p\}$ is a basis for $g^{-1}(W') = W$. Let $C = (C^1, \dots, C^p) \in V$.

Proposition 3.1.4 shows that the element $g \in GL(q, \mathbb{R})$ can be chosen so that $g(W) = W'$ and one of the following conditions is satisfied.

a) $g \in SL(q, \mathbb{R})$.

b) $g = h\alpha$, where $h \in SL(q, \mathbb{R})$ and $\alpha \in GL(q, \mathbb{R})$ with $\det \alpha < 0$.

In the first case $SL(q, \mathbb{R})(C) = SL(q, \mathbb{R})(g^{-1}E) = SL(q, \mathbb{R})(E)$, and in the second case $SL(q, \mathbb{R})(C) = SL(q, \mathbb{R})(\alpha^{-1}(h^{-1}E)) = \alpha^{-1}SL(q, \mathbb{R})(h^{-1}E) = \alpha^{-1}SL(q, \mathbb{R})(E)$. In either case the orbit $SL(q, \mathbb{R})(C)$ is closed in V since $SL(q, \mathbb{R})(E)$ is closed in V . This completes the proof of 3) \Rightarrow 1).

1) \Rightarrow 2). Let $\{C^1, \dots, C^p\}$ be a basis of W such that if $C = (C^1, \dots, C^p) \in V$, then the orbit $SL(q, \mathbb{R})(C)$ is closed in V . Let $\{D^1, \dots, D^p\}$ be any basis of W and let $D = (D^1, \dots, D^p) \in V$. Choose $g \in GL(p, \mathbb{R})$ such that $g(C^i) = D^i$ for $1 \leq i \leq p$. Hence $g(C) = D$.

Since the actions of $GL(q, \mathbb{R})$ and $GL(p, \mathbb{R})$ on V commute by the discussion above it follows that $SL(q, \mathbb{R})(D) = SL(q, \mathbb{R})(g(C)) = g SL(q, \mathbb{R})(C)$ is closed in V since $SL(q, \mathbb{R})(C)$ is closed in V . This proves that 1) \Rightarrow 2).

2) \Rightarrow 4). Let $\{C^1, \dots, C^p\}$ be any basis of W , and let $C = (C^1, \dots, C^p) \in V$. Then $SL(q, \mathbb{R})(C)$ is closed in V by the hypothesis of 2). By the theorem in (7.1) there exists $g \in SL(q, \mathbb{R})$ such $g(C) = D = (D^1, \dots, D^p)$ is a minimal vector for $SL(q, \mathbb{R})$. Since $g(C^i) = D^i$ for $1 \leq i \leq p$ it follows that $\{D^1, \dots, D^p\}$ is a basis for $g(W) = W'$. By Proposition 7.2.1 $\sum_{i=1}^p (D^i)^2$ is a negative multiple of the identity. This completes the proof of 2) \Rightarrow 4). \square

Proposition 7.4.2. Let \mathfrak{N} be a 2-step nilpotent Lie algebra of type (p, q) . Let $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ be an adapted basis of \mathfrak{N} (cf. (3.1)), and define structure matrices $\{C^1, \dots, C^p\}$ in $\mathfrak{so}(q, \mathbb{R})$ by

$$[v_i, v_j] = \sum_{k=1}^p C_{ij}^k Z_k \quad \text{for all } i, j.$$

Let $C_{\mathfrak{B}} = (C^1, \dots, C^p) \in V = \mathfrak{so}(q, \mathbb{R})^p$. Then the following statements are equivalent.

- 1) The orbit $SL(q, \mathbb{R})(C_{\mathfrak{B}})$ is closed in V for some adapted basis \mathfrak{B} .
- 2) The orbit $SL(q, \mathbb{R})(C_{\mathfrak{B}})$ is closed in V for every adapted basis \mathfrak{B} .
- 3) a) $[\mathfrak{N}, \mathfrak{N}]$ is the center of \mathfrak{N} .
b) \mathfrak{N} admits an inner product $\langle \cdot, \cdot \rangle$ whose Ricci tensor is geodesic flow invariant.

Proof. We shall prove 2) \Rightarrow 1), 1) \Rightarrow 3) and 3) \Rightarrow 2). The first of these implications is obvious.

We prove 1) \Rightarrow 3). Let \mathfrak{B} be an adapted basis with structure matrices $\{C^1, \dots, C^p\}$ such that the orbit $SL(q, \mathbb{R})(C_{\mathfrak{B}})$ is closed in V . Let $W = \text{span} \{C^1, \dots, C^p\} \subseteq \mathfrak{so}(q, \mathbb{R})$. Then \mathfrak{N} is Lie algebra isomorphic to the standard, metric, 2-step nilpotent Lie algebra $\mathfrak{N}' = \mathbb{R}^q \oplus W$ by the proof of Proposition 3.1.1. By Proposition 7.4.1 we see that $[\mathfrak{N}', \mathfrak{N}'] = W$ is the center of \mathfrak{N}' , and \mathfrak{N}' admits an inner product whose Ricci tensor is geodesic flow invariant. Hence the same is true for \mathfrak{N} , which is isomorphic to \mathfrak{N}' .

We prove 3) \Rightarrow 2). Let \mathfrak{B} be an adapted basis of \mathfrak{N} with structure matrices $\{C^1, \dots, C^p\} \subseteq \mathfrak{so}(q, \mathbb{R})$, and let $W = \text{span} \{C^1, \dots, C^p\}$. By the proof of Proposition 3.1.1 \mathfrak{N} is isomorphic to the standard, metric, 2-step nilpotent Lie algebra $\mathfrak{N}' = \mathbb{R}^q \oplus W$. By hypothesis $[\mathfrak{N}, \mathfrak{N}]$ is the center of \mathfrak{N} , and \mathfrak{N} admits an inner product $\langle \cdot, \cdot \rangle$ whose Ricci tensor is geodesic flow invariant. Hence $[\mathfrak{N}', \mathfrak{N}'] = W$ is the center of \mathfrak{N}' , and \mathfrak{N}' admits an inner product whose Ricci tensor is geodesic flow invariant. From the implication 3) \Rightarrow 2) in Proposition 7.4.1 we conclude that the orbit $SL(q, \mathbb{R})(C)$ is closed in $V = \mathfrak{so}(q, \mathbb{R})^p$, where $C = (C^1, \dots, C^p) \in V$. \square

7.5. Minimal $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ vectors and optimal Ricci tensor.

Standard 2-step examples with modified inner products on W .

Proposition. *Let W be a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$. Then the following statements are equivalent.*

- 1) W admits a basis $\{C^1, \dots, C^p\}$ such that
 - a) $\sum_{i=1}^p (C^i)^2 = -\lambda^2 \text{Id}$ for some positive number λ .
 - b) $\langle C^i, C^j \rangle = -\text{trace}(C^i C^j) = \mu^2 \delta_{ij}$ for some positive number μ .
- 2) Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be the standard, metric, 2-step nilpotent Lie algebra determined by W . Then
 - a) W is the center of \mathfrak{N} .
 - b) W admits an inner product $\langle \cdot, \cdot \rangle^*$ such that $\{\mathfrak{N}, \langle \cdot, \cdot \rangle^*\}$ has optimal Ricci tensor, where $\langle \cdot, \cdot \rangle^*$ denotes the extension of $\langle \cdot, \cdot \rangle$ to \mathfrak{N} .
- 3) W admits a basis $\{C^1, \dots, C^p\}$ such that $C = (C^1, \dots, C^p)$ is a minimal vector for $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ acting on $V = \mathfrak{so}(q, \mathbb{R})^p = \mathfrak{so}(q, \mathbb{R}) \times \dots \times \mathfrak{so}(q, \mathbb{R})$ (p times).

Proof. The equivalence of 1) and 3) is Proposition 7.2.3. Hence we need only prove the equivalence of 1) and 2).

1) \Rightarrow 2). We need to modify only slightly the proof of 1) \Rightarrow 2) in Proposition 7.3.2. The verification of 2a) is the same. To verify 2b) let $\{C^1, \dots, C^p\}$, $\{\rho_1, \dots, \rho_p\}$, $\langle \cdot, \cdot \rangle^*$ and $j^* : W = \mathfrak{z} \rightarrow \mathfrak{so}(\mathbb{R}^q, \langle \cdot, \cdot \rangle^*)$ have the same meaning as in the proof of 1) \Rightarrow 2) in Proposition 7.3.2. As before it follows that $j^*(\rho_k)$ has matrix $-C^k$ relative to $\{e_1, \dots, e_p\}$ for $1 \leq k \leq p$, and $\sum_{i=1}^p j^*(\rho_k)^2 = -\lambda^2 \text{Id}$, where λ is the positive constant that occurs in 1a). Hence $\mathbf{Ric}(X, Y) = (1/2) \langle \sum_{i=1}^p j^*(\rho_k)^2 X, Y \rangle = -(1/2) \lambda^2 \langle X, Y \rangle$ for all X, Y in \mathbb{R}^q by (1.5).

By (1.5) we also have $\mathbf{Ric}(\rho_i, \rho_k) = -(1/4) \text{trace } j^*(\rho_i) j^*(\rho_k) = -(1/4) \text{trace } C^i C^k = (1/4) \langle C^i, C^k \rangle = (1/4) \mu^2 \delta_{ik}$, where μ is the positive constant that occurs in 1b). Since $\{\rho_1, \dots, \rho_p\}$ is an orthonormal basis for W relative to $\langle \cdot, \cdot \rangle^*$ it follows that $\mathbf{Ric}(Z, Z^*) = (1/4) \mu^2 \langle Z, Z^* \rangle$ for all Z, Z^* in W . By definition this proves that the Ricci tensor of $\{\mathfrak{N}, \langle \cdot, \cdot \rangle^*\}$ is optimal.

2) \Rightarrow 1). The proof is the same as in the proof of 2) \Rightarrow 1) in Proposition 7.3.2 except for slight and obvious changes. \square

7.6. Closed $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ orbits and optimal Ricci tensor.

Proposition 7.6.1. *Let $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ act on $V = \mathfrak{so}(q, \mathbb{R})^p = \mathfrak{so}(q, \mathbb{R}) \times \dots \times \mathfrak{so}(q, \mathbb{R})$ (p times) as defined at the end of (7.1). Let W be a p -dimensional subspace of $\mathfrak{so}(q, \mathbb{R})$. Then the following statements are equivalent.*

- 1) *For some basis $\{C^1, \dots, C^p\}$ of W the orbit $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})(C)$ is closed in V , where $C = (C^1, \dots, C^p)$.*
- 2) *For every basis $\{C^1, \dots, C^p\}$ of W the orbit $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})(C)$ is closed in V , where $C = (C^1, \dots, C^p)$.*
- 3) *Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be the standard, metric, 2-step nilpotent Lie algebra determined by W .*

Then

- a) *W is the center of \mathfrak{N} .*
- b) *\mathfrak{N} admits an inner product $\langle \cdot, \cdot \rangle^*$ such that $\{\mathfrak{N}, \langle \cdot, \cdot \rangle^*\}$ has optimal Ricci tensor.*
- 4) *There exists $g \in SL(q, \mathbb{R})$ such that if $W' = g(W) = gWg^t$, then W' admits a basis $\{D^1, \dots, D^p\}$ such that*

$$\begin{aligned} \text{a) } \sum_{i=1}^p (D^i)^2 &= -\lambda^2 \text{Id} && \text{for some positive number } \lambda. \\ \text{b) } \langle D^i, D^j \rangle &= -\text{trace}(D^i D^j) = \mu^2 \delta_{ij} && \text{for some positive number } \mu. \end{aligned}$$

Proof. This is very similar to the proof of Proposition 7.4.1, but here we use the Proposition in 7.5 instead of Proposition 7.3.2 and Proposition 7.2.3 instead of Proposition 7.2.1. We omit the details. \square

Proposition 7.6.2. *Let \mathfrak{N} be a 2-step nilpotent Lie algebra of type (p, q) . Let $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ be an adapted basis of \mathfrak{N} (cf. (3.1)), and define structure matrices $\{C^1, \dots, C^p\}$ in $\mathfrak{so}(q, \mathbb{R})$ by*

$$[v_i, v_j] = \sum_{k=1}^p C_{ij}^k Z_k \quad \text{for all } i, j.$$

Let $C_{\mathfrak{B}} = (C^1, \dots, C^p) \in V = \mathfrak{so}(q, \mathbb{R})^p$. Then the following statements are equivalent.

- 1) *The orbit $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})(C_{\mathfrak{B}})$ is closed in V for some adapted basis \mathfrak{B} .*
- 2) *The orbit $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})(C_{\mathfrak{B}})$ is closed in V for every adapted basis \mathfrak{B} .*
- 3) a) *$[\mathfrak{N}, \mathfrak{N}]$ is the center of \mathfrak{N} .*
 b) *\mathfrak{N} admits an inner product $\langle \cdot, \cdot \rangle$ whose Ricci tensor is optimal.*

Proof. This proof also is very similar to the proof of Proposition 7.4.2, but here we use Proposition 7.6.1 instead of Proposition 7.4.1. We omit the details. \square

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