Canonical inner products on $C\ell(n)$ and M(n,K)

Let $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let M(n,K) denote the n x n matrices with elements in K. Let $C\ell(n)$ denote the Clifford algebra determined by $\mathbb{R}^n$ with the canonical inner product $< , >$. We first describe canonical inner products on both $C\ell(n)$ and M(n,K). We then show that if $\varphi : C\ell(n) \to M(p,K)$ is an algebra isomorphism such that $\varphi(x)^* = \varphi(\overline{x})$ for all $x \in C\ell(n)$, then φ is a linear isometry with respect to the canonical inner products on $C\ell(n)$ and M(p,K).

Elsewhere, we show that for every positive integer $n \neq 4k+3$ there exists a positive integer p and an algebra isomorphism $\varphi : C\ell(n) \to M(p,K)$ for $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If $n = 4k+3$, then $C\ell(n) = A_1 \oplus A_2$, where A_1 and A_2 are two sided commuting ideals of $C\ell(n)$ that are both algebra isomorphic to $C\ell(n-1) \approx M(p,K)$. In this case $A_1 \oplus A_2$ and $M(p,K) \oplus M(p,K)$ both have canonical inner products induced from the factors, and there exists an algebra isomorphism $\varphi : C\ell(n) = A_1 \oplus A_2 \to M(p,K) \oplus M(p,K)$ such that $\varphi(x)^* = \varphi(\overline{x})$ for all $x \in C\ell(n)$ and φ is also a linear isometry with respect to these inner products.

Canonical inner products on $C\ell(n)$

Proposition 1 There exists a unique inner product $< , >$ such that

1) $< 1, 1 > = 1$.

2) $L(x)^* = L(\overline{x})$ for all $x \in C\ell(n)$.

2) $R(x)^* = R(\overline{x})$ for all $x \in C\ell(n)$.

Remark L(x) and R(x) denote left and right translation by x, and * denotes the metric transpose operation.

Proof (Existence) Define $< x, y > = \operatorname{Re} x\,\overline{y}$ for all $x,y \in C\ell(n)$. We show that $< , >$ is an inner product that satisfies 1), 2) and 3). We begin by showing that $< , >$ is an inner product.

Real part of an element of $C\ell(n)$

Let $\{e_1, \dots , e_n\}$ denote the standard orthonormal basis of $\mathbb{R}^n$. Let $\mathfrak{I}_k$ denote the set of multi-indices $I = (i_1, \dots , i_k)$, where $i_1 < i_2 < \dots < i_k$. For each multi-index $I = (i_1, \dots , i_k)$ in $\mathfrak{I}_k$ let e_I denote the product $e_{i_1} \cdot e_{i_2} \cdot \dots \cdot e_{i_k}$. Recall that a vector space basis for $C\ell(n)$ is given by $\mathfrak{B} = \{1, e_I : I \in \mathfrak{I}_k, 1 \leq k \leq n\}$. For an element x of $C\ell(n)$ we write $x = a + \Sigma\, a_I\, e_I$, where the sum is over all multi-indices $I \in \mathfrak{I}_k$, $1 \leq k \leq n$, and a, a_I are real numbers. The leading term a we call the <u>real part</u> of x, denoted Re x. Note that Re x would be the same if we had chosen a different orthonormal basis $\{e_1 \dots , e_n\}$ of $\mathbb{R}^n$ to define the multiplication in $C\ell(n)$.

Lemma 1 Re xy = Re yx for all x,y in $C\ell(n)$.

Proof If $x = a + \Sigma\, a_I\, e_I$ and $y = b + \Sigma\, b_I\, e_I$, then Re xy = Re $(ab + \Sigma\, a_I\, b_J\, e_I\, e_J)$. By inspection $e_I\, e_J = \pm\, e_K$ for some multi-index K unless I = J. If I = J, then $e_I\, e_J = e_I^{\ 2} = \pm 1$. Hence Re $xy = ab + \Sigma\, a_I\, b_I\, e_I^{\ 2}$, where the sum is over all multi-indices $I \in \mathfrak{J}_k$, $1 \le k \le n$. This expression is symmetric in the components of x and y, so we obtain Re xy = Re yx.

Lemma 2 $< , >$ is an inner product on $C\ell(n)$ for which the standard basis $\mathfrak{B}$ above is orthonormal. The canonical automorphism α preserves this inner product.

Proof If $x = a + \Sigma\, a_I\, e_I$ and $y = b + \Sigma\, b_I\, e_I$, then Re $x\,\bar{y}$ = Re $(ab + \Sigma\, a_I\, b_J\, e_I\, \overline{e_J}\,)$. As above $e_I\, \overline{e_J} = \pm\, e_K$ for some multi-index K unless I = J. If I = J, then $e_I\, \overline{e_I} = N(e_I) = 1$. Hence Re $x\,\bar{y} = ab + \Sigma\, a_I\, b_I$. This is clearly a symmetric, positive definite $\mathbb{R}-$ bilinear form, and by inspection the basis $\mathfrak{B}$ is orthonormal.

It remains only to check that α preserves the inner product. Note that Re $\alpha(x)$ = Re x for all $x \in C\ell(n)$ since α fixes the real numbers. Let $x,y \in C\ell(n)$ be given. Then $< \alpha(x), \alpha(y) > = \text{Re}\ \alpha(x)\, \overline{\alpha(y)} = \text{Re}\ \alpha(x)\, \alpha(\bar{y}) = \text{Re}\ \alpha(x\, \bar{y}) = \text{Re}\ x\, \bar{y} = < x, y >$.□

Next we verify that properties 1), 2) and 3) of the Proposition are satisfied by $< , >$.

1) It is obvious from the definition that $< 1, 1 > = 1$.

For 2) and 3) let elements x,y and $z \in C\ell(n)$ be given.

2) $< L(x)\, y , z > = < xy, z > = \text{Re}\ x\, y\, \bar{z}$. On the other hand $< y, L(\bar{x})\, z > = < y, \bar{x}\, z > = \text{Re}\ y\, \overline{(\bar{x}\, z)} = \text{Re}\ y\, \bar{z}\, x = \text{Re}\ x\, y\, \bar{z}$ by Lemma 1.

3) $< R(x)\, y , z > = < yx, z > = \text{Re}\ y\, x\, \bar{z}$, and $< y, R(\bar{x})\, z > = < y, z\, \bar{x} > = \text{Re}\ y\, \overline{(z\, \bar{x})} = \text{Re}\ y\, x\, \bar{z}$.

(Uniqueness) We now prove the uniqueness of the inner product $< , >$ satisfying properties 1), 2) and 3). We let $< , >_1$ denote the inner product defined above with $< x, y > = \text{Re}\ x\, \bar{y}$ for all $x,y \in C\ell(n)$. Let $< , >_2$ be any other inner product on $C\ell(n)$ that satisfies 1), 2) and 3). Since $< , >_1$ is nondegenerate there exists a nonsingular linear transformation $S : C\ell(n) \to C\ell(n)$ such that $< x, y >_2 = \ < Sx, y >_1$ for all $x,y \in C\ell(n)$. It is routine to check that S is positive definite and symmetric with respect to both $< , >_1$ and $< , >_2$. We show that S is the identity.

Lemma 3 $x\, S(y) = S(xy) = S(x)\, y$ for all $x,y \in C\ell(n)$.

Proof Let arbitrary elements x,y and $z \in C\ell(n)$ be given. Then $< S(xy), z >_1 = < xy , z >_2$ $= < R(y)\, x , z >_2 = < x, R(\bar{y})\, z >_2 = < S(x), R(\bar{y})\, z >_1 = < R(y)\, S(x) , z >_1 = < S(x)\, y , z >_1$. This proves

(1) $S(xy) = S(x)\, y$ for all $x,y \in C\ell(n)$.

Next, $< S(xy), z >_1 = < xy , z >_2 = < L(x)\, y , z >_2 = < y, L(\bar{x})\, z >_2 = < S(y), L(\bar{x})\, z >_1 =$ $< L(x)\, S(y) , z >_1 = < x\, S(y) , z >_1$, which proves

(*2*) $S(xy) = x\, S(y)$ for all $x,y \in C\ell(n)$.

The proof of the lemma is complete.□

Lemma 4 If $\alpha = S(1)$, then α lies in the center of $C\ell(n)$, and $S(x) = \alpha x = x\alpha$ for all $x \in C\ell(n)$.

Proof We note that $S(y) = \alpha y$ for all $y \in C\ell(n)$ by substituting $x = 1$ in equation (1) of the proof of Lemma 3. Similarly, we observe that $S(x) = x\alpha$ for all $x \in C\ell(n)$ by substituting $y = 1$ in equation (2) of the proof of Lemma 3.□

We now complete the proof of the uniqueness assertion of the Proposition by showing that $\alpha = 1$. By the hypotheses and Lemma 4 we have $1 = < 1, 1 >_2 = < S(1), 1 >_1 = < \alpha , 1 > = \mathrm{Re}\ \alpha$.

Next we show that $\alpha = \bar{\alpha}$. Let elements $x,y \in C\ell(n)$ be given arbitrarily. Note that $< x, y >_2 = < S(x), y >_1 = < \alpha\, x, y >_1 = \mathrm{Re}\ \alpha\, x\, \bar{y}$. Similarly, $< x, y >_2 = < y, x >_2 =$ $\mathrm{Re}\ \alpha\, y\, \bar{x} = \mathrm{Re}\ \overline{(\alpha\, y\, \bar{x})} = \mathrm{Re}\ x\, \bar{y}\, \bar{\alpha} = \mathrm{Re}\ \bar{\alpha}\, x\, \bar{y}$. Hence $< \alpha\, x , y >_1 = \mathrm{Re}\ \alpha\, x\, \bar{y} = < x, y >_2$ $= \mathrm{Re}\ \bar{\alpha}\, x\, \bar{y} = < \bar{\alpha}\, x , y >_1$ for all $x,y \in C\ell(n)$. It follows that $\alpha\, x = \bar{\alpha}\, x$ for all $x \in C\ell(n)$, and we conclude that $\alpha = \bar{\alpha}$ by setting $x = 1$.

Finally, we show that $\alpha = 1$ by analyzing the center of $C\ell(n)$. The case $n = 4k+3$ requires special consideration. In the discussion below we refer to results in the handout " Two sided ideals in Clifford algebras".

If $n = 4k$ or $4k+2$, then by Proposition 1 and its proof the center of $C\ell(n)$ is $\mathbb{R}$. In this case it is clear that $\alpha = 1$ since $\mathrm{Re}\ \alpha = 1$.

To describe the center of $C\ell(n)$ when n is odd we need to consider the central element $z = e_1 \cdot e_2 \ldots \cdot e_n \in C\ell(n)$. If $n = 4k+1$ then the center of $C\ell(n)$ consists of $\{a + bz : a,b \in \mathbb{R}\}$ by Lemma 1b and the proof of Proposition 1. It is not difficult to show that $\bar{z} = -z$. Hence if $\alpha = a + bz$ for suitable $a,b \in \mathbb{R}$, then $\bar{\alpha} = a - bz$. Since $\alpha = \bar{\alpha}$ it follows that $b = 0$. Since $a = \mathrm{Re}\ \alpha = 1$ we conclude that $\alpha = 1$.

We now consider the case $n = 4k+3$. Here the center of $C\ell(n)$ also consists of $\{a + bz : a,b \in \mathbb{R}\}$ by Lemma 1b and the proof of Proposition 1. Moreover, Proposition 1 shows that $z^2 = 1$. However, in this case it is not difficult to show that $\bar{z} = z$, and this means we cannot apply the argument used in the case $n = 4k+1$. Instead we use Proposition 2 to complete the proof when $n = 4k+3$.

In the case $n = 4k+3$ $C\ell(n)$ admits exactly two proper two sided ideala A_1 and A_2. Both are isomorphic as algebras to $C\ell(n-1)$ and $C\ell(n) = A_1 \oplus A_2$. Moreover, if $e = (1/2)(1-z)$ and $e' = (1/2)(1+z)$ then $A_1 = e\, C\ell(n)$ and $A_2 = e'\, C\ell(n)$. Note that $xy = yx = 0$ if $x \in A_1$ and $y \in A_2$ since A_1 and A_2 are disjoint two sided ideals in $C\ell(n)$. Moreover A_1 and A_2 are closed under conjugation since $\bar{e} = e$ and $\bar{e'} = e'$. We note that $L(x)$ and $L(x)^* = L(\bar{x})$ leave A_1 invariant and are zero on A_2 for all $x \in A_1$. Similarly, $L(y)$ and $L(y)^* = L(\bar{y})$ leave A_2 invariant and are zero on A_1 for all $y \in A_2$.

It suffices to prove that A_1 and A_2 are orthogonal with respect to both $< , >_1$ and $< , >_2$. It then follows that $< , >_1 = < , >_2$ on both A_1 and A_2 by the discussion above for the case $n = 4k+2$. This will complete the proof in the case $n = 4k+3$.

We write arbitrary elements of A_1 and A_2 as $e\,x$ and $e'\,y$, where x and y are arbitrary elements of $C\ell(4k+3)$. Since $\bar{z} = z$ and $z^2 = 1$ we obtain $< e\,x \;\; e'\,y >_1 =$ $(1/4) < (1-z)\,x\,,\,(1+z)\,y >_1 = (1/4)\,\{ < x\,,\,y >_1 + < x\,,\,zy >_1 - < zx\,,\,y >_1 - < zx\,,\,zy >_1 \} =$ $(1/4)\,\{ < x\,,\,y >_1 + < \bar{z}\,x\,,\,y >_1 - < zx\,,\,y >_1 - < \bar{z}\,zx\,,\,y >_1 \} = 0$. In proving that A_1 and A_2 are orthogonal with respect to $< , >_1$ we used only the fact that $L(z)^* = L(\bar{z})$, which holds also for $< , >_2$ by hypothesis. Hence A_1 and A_2 are orthogonal with respect to $< , >_2$, and the proof of the Proposition in the case $n = 4k+3$ is complete.□

Canonical inner products on M(n,K)

Let $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $M(n,K)$ denote the $n \times n$ matrices with elements in K. Let $*$ denote the conjugate transpose operation on $M(n,K)$, and also the metric transpose operation on $M(n,K)$ with respect to an inner product $< , >$. Let $L(A)$ and $R(A)$ denote left and right translation by $A \in M(n,K)$.

Proposition 2 There exists a unique inner product $< , >$ on $M(n,K)$ such that

1) $< 1, 1 > = 1$.
2) $L(A)^* = L(A^*))$ for all $A \in M(n,K)$.
3) $R(A)^* = R(A^*)$ for all $A \in M(n,K)$.

We shall consider only the hardest case when $K = \mathbb{H}$. The proofs in the other two cases are essentially identical but easier since $\mathbb{R}$ and $\mathbb{C}$ are commutative.

Existence of < , >

Define $< A, B > = (1/n)\ \mathrm{Re\ trace}\ (AB^*)$. We show that $< , >$ satisfies 1), 2) and 3). This definition makes sense for all of the cases $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Lemma 1 If $x,y \in \mathbb{H}$, then Re xy = Re yx.

Proof Let $x = x_o + x_1 i + x_2 j + x_3 k$ and $y = y_o + y_1 i + y_2 j + y_3 k$. Then Re xy = $x_o y_o - x_1 y_1 - x_2 y_2 - x_3 y_3$ = Re yx.□

Lemma 2 If $A,B \in M(n,\mathbb{H})$, then Re trace (AB) = Re trace (BA).

Proof Trace (AB) = $\sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ki} A_{ik}$, and similarly trace (BA) = $\sum_{i=1}^{n} \sum_{k=1}^{n} A_{ki} B_{ik} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$, interchanging i and k. Since Re $(B_{ki} A_{ik})$ = Re $(A_{ik} B_{ki})$ by lemma 1 it follows that Re trace (AB) = Re trace (BA).□

We now are ready to prove that $< , >$ satisfies 1), 2) and 3). We show first that $< , >$ is a positive definite inner product on $M(n,\mathbb{H})$. Clearly, $< , >$ is $\mathbb{R}$–bilinear. We show first that $< , >$ is symmetric. If $A,B \in M(n,\mathbb{H})$, then $< A, B > = (1/n)$ Re trace (AB^*) = $(1/n)\ \mathrm{Re}\ (\sum_{i=1}^{n} (AB^*)_{ii}) = (1/n)\ \mathrm{Re}\ (\sum_{i=1}^{n} \sum_{k=1}^{n} B^*_{ki} A_{ik}) = (1/n)\ \mathrm{Re}\ (\sum_{i=1}^{n} \sum_{k=1}^{n} \overline{B_{ik}} A_{ik})$ = $(1/n)\ \mathrm{Re}\ (\sum_{i=1}^{n} \sum_{k=1}^{n} \overline{(\overline{B_{ik}} A_{ik})}) = (1/n)\ \mathrm{Re}\ (\sum_{i=1}^{n} \sum_{k=1}^{n} \overline{A_{ik}} B_{ik}) = < B, A >$. Finally, $< , >$ is positive definite since $< A, A > = (1/n)\ \mathrm{Re}\ (\sum_{i=1}^{n} \sum_{k=1}^{n} \overline{A_{ik}} A_{ik}) = \mathrm{Re}\ (\sum_{i=1}^{n} \sum_{k=1}^{n} |A_{ik}|^2) \geq 0$ with equality ⇔ $A_{ik} = 0$ for all i,k.

Assertion 1) of the Proposition clearly holds. We prove 2) and 3). Let A,B and C $\in M(n,\mathbb{H})$ be given. We compute $< B, L(A)^*C > = < L(A)B, C > = < AB, C >$ = $(1/n)$ Re trace (ABC^*). Next, $< B, L(A^*)C > = < B, A^*C > = (1/n)$ Re trace $B(A^*C)^*$ = $(1/n)$ Re trace $BC^*A = (1/n)$ Re trace $ABC^* = < B, L(A)^*C >$ by Lemma 2. Hence $L(A)^* = L(A^*)$ for all $A \in M(n,\mathbb{H})$, which proves 2).

Similarly, $< B, R(A)^*C > = < R(A)B, C > = < BA, C > = (1/n)$ Re trace BAC^* while $< B, R(A^*)C > = < B, CA^* > = (1/n)$ Re trace $B(CA^*)^* = (1/n)$ Re trace BAC^* = $< B, R(A)^*C >$. Hence $R(A)^* = R(A^*)$ for all $A \in M(n,\mathbb{H})$, which proves 3).

Uniqueness of < , >

Define the inner product $< , >_1$ on $M(n,\mathbb{H})$ by $< A, B >_1 = (1/n)$ Re trace AB^*. We showed above that $< , >_1$ satisfies properties 1), 2) and 3) of the proposition. Next, suppose that $< , >_2$ is another inner product on $M(n,\mathbb{H})$ that satisfies properties 1), 2) and 3). Write $< A, B >_2 = < S(A), B >_1$ for all $A,B \in M(n,\mathbb{H})$, where $S : M(n,\mathbb{H}) \rightarrow M(n,\mathbb{H})$ is $\mathbb{R}$–linear and invertible. It is easy to see that S is symmetric and positive definite with respect to both $< , >_1$ and $< , >_2$.

Lemma 1 $AS(B) = S(AB) = S(A)B$ for all $A,B \in M(n,\mathbb{H})$.

Proof Let $A, B, C \in M(n,\mathbb{H})$. Since property 2) holds for both $<\,,\,>_1$ and $<\,,\,>_2$ we have $< AS(B), C >_1 = < S(B), A^*C >_1 = < B, A^*C >_2 = < AB, C >_2 = < S(AB), C >_1$. This proves that $AS(B) = S(AB)$ for all $A,B \in M(n,\mathbb{H})$.

Since property 3) holds for both $<\,,\,>_1$ and $<\,,\,>_2$ we have $< S(A)B, C >_1 = < R(B)S(A), C >_1 = < S(A), R(B^*)C >_1 = < A, R(B^*)C >_2 = < R(B)A, C >_2 = < AB, C >_2 = < S(AB), C >_1$. This proves that $S(A)B = S(AB)$ for all $A,B \in M(n,\mathbb{H})$.□

Lemma 2 Let $\alpha = S(Id)$. Then $\alpha = \lambda\, Id$ and $S = \lambda\, Id$ for some real number λ.

Proof Using the equality $AS(B) = S(AB)$ and setting $B = Id$ yields $A\alpha = S(A)$ for all $A \in M(n,\mathbb{H})$. Using the equality $S(AB) = S(A)B$ and setting $A = Id$ yields $S(B) = \alpha B$ for all $B \in M(n,\mathbb{H})$. This shows that α lies in the center of $M(n,\mathbb{H})$, which equals $\mathbb{R} Id$. If $\alpha = \lambda\, Id$ for some real number λ, then $S = \lambda\, Id$ since $S(A) = \alpha A$ for all $A \in M(n,\mathbb{H})$ by the discussion above.□

We now show that $<\,,\,>_1 = <\,,\,>_2$. Since $<\,,\,>_1$ and $<\,,\,>_2$ both satisfy 1) we use Lemma 2 to conclude $1 = < Id, Id >_2 = < S(Id), Id >_1 = \lambda < Id, Id >_1 = \lambda$. Hence $S = Id$, which proves that $<\,,\,>_1 = <\,,\,>_2$.

Proposition Let $\varphi : C\ell(n) \to M(p,K)$ be an algebra isomorphism for $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ such that $\varphi(x)^* = \varphi(\overline{x})$ for all $x \in C\ell(n)$. Let $<\,,\,>$ and $<\,,\,>^*$ be the canonical inner products on $C\ell(n)$ and $M(p,K)$ defined by $< x, y > = \mathrm{Re}\; x\overline{y}$ for all $x,y \in C\ell(n)$ and $< A, B >^* = (1/n)\, \mathrm{Re\ trace}\; AB^*$ for all $A,B \in M(p,K)$. Then $< \varphi(x), \varphi(y) >^* = < x, y >$ for all $x,y \in C\ell(n)$.

Proof Let $<\,,\,>$ denote the canonical inner product on $C\ell(n)$, and let $<\,,\,>'$ be the inner product on $M(p,K)$ such that $< \varphi(x), \varphi(y) >' = < x, y >$ for all $x,y \in C\ell(n)$. We assert that $<\,,\,>^* = <\,,\,>'$, By Proposition 2 it suffices to show that $<\,,\,>'$ satisfies the three conditions stated in Proposition 2. Clearly $< Id, Id >' = 1$, so condition 1) holds.

Let A,B,C be elements of $M(p,K)$, and let x,y,z be those elements of $C\ell(n)$ such that $\varphi(x) = A$, $\varphi(y) = B$ and $\varphi(z) = C$.

We verify condition 2) : $L(A)^* = L(A^*)$ for all $A \in M(p,K)$. We compute $< L(A)B, C >' = < AB, C >' = < \varphi(xy), \varphi(z) >' = < xy, z > = < y, \overline{x}z > = < \varphi(y), \varphi(\overline{x})\varphi(z) >' = < \varphi(y), \varphi(x)^*\varphi(z) >' = < B, A^*C >' = < B, L(A^*)C >'$.

We verify condition 3) : $R(A)^* = R(A^*)$ for all $A \in M(p,K)$. We compute $< R(A)B, C >' = < BA, C >' = < \varphi(yx), \varphi(z) >' = < yx, z > = < y, z\overline{x} > = < \varphi(y), \varphi(z)\varphi(\overline{x}) >' = < \varphi(y), \varphi(z)\varphi(x)^* >' = < B, CA^* >' = < B, R(A^*)C >'$.

We conclude that $<\,,\,>^* = <\,,\,>'$ since $<\,,\,>'$ satisfies the three conditions stated in Proposition 2. The proof of the Proposition is complete. □