Math 33A Fall 2005 Homework Assignments Eberlein

HW # 8 due Wednesday November 2

11.5 # 1, 3, 5, 7, 9, 17, 19, 23, 25, 27, 33
11.6 # 7, 8, 11, 13, 19, 21, 35, 37

Extra Problems

1. (Laplace's equation in polar coordinates) Consider a function f(x,y), which becomes a function of the polar coordinates r,θ when we substitute x = r cos θ and y = r sin θ. Prove :

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

One denotes either side of the expression above by Δf, and the equation Δf = 0 is called Laplace's equation.

(Hint : Use the following formulas from class and the handout :

$$\frac{\partial}{\partial x} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial \theta}$$

2. Let f(r,θ) be a function in polar coordinates of the form f(r,θ) = A(r) B(θ) ; that is, a product of a function A(r) of r and a function B(θ) of θ. Because of the nature of polar coordinates we impose the additional condition that B(θ + 2π) = B(θ) for all real numbers θ.

a) Show that if f satisfies Laplace's equation ; that is, Δf = 0, then

$$\{A''(r) + \frac{1}{r}A'(r) + \frac{1}{r^2}A(r)\frac{B''(\theta)}{B(\theta)}\}\, B(\theta) = 0.$$

Conclude that $$\frac{B''(\theta)}{B(\theta)} = -\frac{r^2 A''(r)}{A(r)} - \frac{rA'(r)}{A(r)}$$

b) Explain why both sides of the equation above must be equal to a constant c.

c) Conclude that B"(θ) - c B(θ) = 0 and r^2 A"(r) + r A'(r) + c A(r) = 0. The second equation is called Bessel's equation,

Remark One can show that the constant c above must equal –1 since B(θ + 2π) = B(θ) for all real numbers θ. In this case the solutions of B"(θ) + B(θ) = 0 are given by B(θ) = a cos θ + b sin θ, where a,b are arbitrary real numbers. It is easy to see that these functions B(θ) satisfy the equation B"(θ) + B(θ) = 0, and the theory of differential equations (Math 83) says that there are no other solutions. How to solve the Bessel's equation $r^2 A''(r) + r A'(r) + c A(r) = 0$ is explained in Math 124.

3. (spherical coordinates) Recall that the spherical coordinates ρ,θ and φ are defined by $\rho^2 = x^2 + y^2 + z^2$, $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$ and $z = \rho \cos\varphi$. Using methods similar to those used in the handout on polar coordinates prove the following formulas :

a) $\dfrac{\partial\rho}{\partial x} = \sin\varphi \cos\theta \qquad \dfrac{\partial\rho}{\partial y} = \sin\varphi \sin\theta \qquad \dfrac{\partial\rho}{\partial z} = \cos\varphi$

b) $\dfrac{\partial\varphi}{\partial x} = \dfrac{\cos\theta \cos\varphi}{\rho} \qquad \dfrac{\partial\varphi}{\partial y} = \dfrac{\sin\theta \cos\varphi}{\rho} \qquad \dfrac{\partial\varphi}{\partial z} = \dfrac{-\sin\varphi}{\rho}$

c) $\dfrac{\partial\theta}{\partial x} = \dfrac{-\sin\theta}{\rho \sin\varphi} \qquad \dfrac{\partial\theta}{\partial y} = \dfrac{\cos\theta}{\rho \sin\varphi} \qquad \dfrac{\partial\theta}{\partial z} = 0$

For a function $f(x,y,z)$ we can also regard f as a function of the spherical coordinates ρ,θ and φ. Use the formulas above and the chain rule to derive the formulas below.

d) $\dfrac{\partial f}{\partial x} = \sin\varphi \cos\theta \dfrac{\partial f}{\partial \rho} - \dfrac{\sin\theta}{\rho \sin\varphi} \dfrac{\partial f}{\partial \theta} + \dfrac{\cos\theta \cos\varphi}{\rho} \dfrac{\partial f}{\partial \varphi}$

e) $\dfrac{\partial f}{\partial y} = \sin\varphi \sin\theta \dfrac{\partial f}{\partial \rho} + \dfrac{\cos\theta}{\rho \sin\varphi} \dfrac{\partial f}{\partial \theta} + \dfrac{\sin\theta \cos\varphi}{\rho} \dfrac{\partial f}{\partial \varphi}$

e) $\dfrac{\partial f}{\partial z} = \cos\varphi \dfrac{\partial f}{\partial \rho} - \dfrac{\sin\varphi}{\rho} \dfrac{\partial f}{\partial \varphi}$