

HW # 3

due Wednesday September 21

9.5 # 3, 5, 6, 10, 19, 21, 27, 47

9.6 # 5,7,13,21,22

**Extra Problems**

1. Let  $A, B$  be  $3 \times 3$  matrices. Show that  $(AB)^t = B^t A^t$ , where  $X^t$  denotes the transpose of  $X$ .  
 (Hint : If  $X_{ij}$  denotes the entry of  $X$  in row  $i$  and column  $j$ , then by definition of the matrix transpose we see that  $X_{ij}^t = X_{ji}$ . Now compute  $(AB)_{ij}^t$  and  $(B^t A^t)_{ij}$  from the definition of the  $ij$  entry of a product of matrices and show that these entries are equal for all  $i, j$ . Question : Does your proof work for  $n \times n$  matrices, not just  $3 \times 3$  matrices ?)

2. Let  $x, y$  be elements of  $\mathbb{R}^3$  regarded as  $3 \times 1$  matrices (column vectors). Let  $A$  be a  $3 \times 3$  matrix, and let  $A^t$  denote its transpose matrix. Show that  $Ax \cdot y = x \cdot A^t y$  for all  $x, y$  in  $\mathbb{R}^3$ , where  $\cdot$  denotes the dot product, and  $Ax, A^t y$  denote matrix products.  
 (Hint : Recall that  $x \cdot y = y^t x = x^t y$ , matrix products, for all  $x, y$  in  $\mathbb{R}^3$ . Use this fact and the result from problem 1 to evaluate  $Ax \cdot y$  and  $x \cdot A^t y$ .)

3. Let  $x, y$  be elements of  $\mathbb{R}^3$  regarded as  $3 \times 1$  matrices (column vectors). Let  $A$  be a  $3 \times 3$  matrix, and let  $A^t$  denote its transpose matrix. Show that  $(*) Ax \cdot Ay = x \cdot y$  for all  $x, y$  in  $\mathbb{R}^3 \Leftrightarrow A^t A = \text{Id}$  (the identity matrix)  $\Leftrightarrow$  the columns of  $A$  are orthogonal to each other and have length one, regarded as vectors in  $\mathbb{R}^3$ .

**Remark :** A matrix  $A$  satisfying  $(*)$  is said to preserve dot products. Matrices  $A$  satisfying  $(*)$  are called **orthogonal** matrices.

(Hint : Use the result of problem 2 to prove the first equivalence. You will also need the fact that if  $x$  is a vector in  $\mathbb{R}^3$  such that  $x \cdot y = 0$  for all  $y$  in  $\mathbb{R}^3$ , then  $x = 0$  (explain why). To prove the second equivalence observe that the statement  $A^t A = \text{Id}$  is equivalent to saying that  $(A^t A)_{ij} = 0$  if  $i \neq j$  and 1 if  $i = j$ . These conditions on the entries of  $A^t A$  say something about the dot products of the columns of  $A$ . Explain.)

4. a) Suppose  $A$  is a  $3 \times 3$  matrix such that  $A^t A = \text{Id}$ , and let  $C_1, C_2, C_3$  denote the columns of  $A$ . If we know the first two columns  $C_1$  and  $C_2$ , then how many choices are there for the remaining column  $C_3$  ?

b) Find a  $3 \times 3$  matrix  $A$  such that  $A^t A = \text{Id}$  and  $\begin{bmatrix} \frac{3}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$  is the first column of  $A$ .

(Hint : In part a) use a property of cross products and the result of problem 3. In part b) use the result of problem 3. There are many correct answers in part b). )