

**CHARACTERS OF SIMPLYLACED NONCONNECTED
GROUPS VERSUS CHARACTERS OF
NONSIMPLYLACED CONNECTED GROUPS**

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ABSTRACT. Let G be a connected, simply-connected, almost simple semisimple group over \mathbf{C} of simplylaced type and let σ be a nontrivial diagram automorphism of G . Let $G\langle\sigma\rangle$ be the (disconnected) group generated by G and σ . As a consequence of a theorem of Jantzen the character of an irreducible representation of $G\langle\sigma\rangle$ (also irreducible on G) on $G\sigma$ can be expressed in terms of a character of an irreducible representation of a certain connected simply connected semisimple group G_σ of nonsimplylaced type. We show how Jantzen's theorem can be deduced from properties of the canonical bases.

Let G be a connected, simply-connected, almost simple algebraic group of simplylaced type over \mathbf{C} . Let T be a maximal torus of G . Let $x_i : \mathbf{C} \rightarrow G$, $y_i : \mathbf{C} \rightarrow G$ ($i \in I$) be homomorphisms which together with T form a pinning (épinglage) of G . We fix a nontrivial automorphism σ of G such that $\sigma(T) = T$, and such that for some permutation $i \mapsto \tilde{i}$ of I we have $\sigma(x_i(a)) = x_{\tilde{i}}(a)$, $\sigma(y_i(a)) = y_{\tilde{i}}(a)$ for all $a \in \mathbf{C}$. For $i \in I$ we write $\sigma(i) = \tilde{i}$. Let $\langle\sigma\rangle$ be the finite subgroup of the automorphism group of G generated by σ and let $G\langle\sigma\rangle$ be the semidirect product of G with $\langle\sigma\rangle$.

Let X be the group of characters $T \rightarrow \mathbf{C}^*$; let Y be the group of one parameter subgroups $\mathbf{C}^* \rightarrow T$ and let $\langle, \rangle : Y \times X \rightarrow \mathbf{Z}$ be the standard pairing. For $i \in I$ we define $\alpha_i \in X$ by $x_i(\alpha_i(t)) = tx_i(1)t^{-1}$, $y_i(\alpha_i(t)^{-1}) = ty_i(1)t^{-1}$ for all $t \in T$. This is a root of G . Let $\check{\alpha}_i \in Y$ be the corresponding coroot. Note that

$$(a) (Y, X, \langle, \rangle, \check{\alpha}_i, \alpha_i (i \in I))$$

is the root datum of G . Now σ induces automorphisms of X, Y denoted again by σ ; these are compatible with \langle, \rangle and we have $\sigma(\alpha_i) = \alpha_{\sigma(i)}$, $\sigma(\check{\alpha}_i) = \check{\alpha}_{\sigma(i)}$ for $i \in I$. Let $X^+ = \{\lambda \in X; \langle \check{\alpha}_i, \lambda \rangle \in \mathbf{N} \forall i \in I\}$.

We set $Y_\sigma = Y/(\sigma - 1)Y$, ${}^\sigma X = \{\lambda \in X; \sigma(\lambda) = \lambda\}$. Note that $\langle, \rangle : Y \times X \rightarrow \mathbf{Z}$ induces a perfect pairing $Y_\sigma \times {}^\sigma X \rightarrow \mathbf{Z}$ denoted again by \langle, \rangle . Let I_σ be the set of σ -orbits on I . For any $\mathcal{O} \in I_\sigma$ let $\check{\alpha}_\mathcal{O} \in Y_\sigma$ be the image of $\check{\alpha}_i$ under $Y \rightarrow Y_\sigma$ where i is any element of \mathcal{O} . Since $\{\check{\alpha}_i; i \in I\}$ is a \mathbf{Z} -basis of Y we see that $\{\check{\alpha}_\mathcal{O}; \mathcal{O} \in I_\sigma\}$ is a \mathbf{Z} -basis of Y_σ . For any $\mathcal{O} \in I_\sigma$ let $\alpha_\mathcal{O} = 2^h \sum_{i \in \mathcal{O}} \alpha_i \in {}^\sigma X$ where h is the

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number of unordered pairs (i, j) such that $i, j \in \mathcal{O}$, and $\alpha_i + \alpha_j$ is a root. Note that $h = 0$ except when G is of type A_{2n} when $h = 0$ for all \mathcal{O} but one and $h = 1$ for one \mathcal{O} . Note that

$$(b) (Y_\sigma, {}^\sigma X, \langle, \rangle, \check{\alpha}_\mathcal{O}, \alpha_\mathcal{O} (\mathcal{O} \in I_\sigma))$$

is a root datum, see [Ja, p.29]. Let ${}^\sigma X^+ = \{\lambda \in {}^\sigma X; \langle \check{\alpha}_\mathcal{O}, \lambda \rangle \in \mathbf{N} \forall \mathcal{O} \in I_\sigma\} = {}^\sigma X \cap X^+$. Let G_σ be the connected semisimple group over \mathbf{C} with root datum (b). By definition, G_σ is provided with an épinglage $(T_\sigma, x_\mathcal{O}, y_\mathcal{O} \quad (\mathcal{O} \in I_\sigma))$ where $T_\sigma := \mathbf{C}^* \otimes Y_\sigma = T / \{\sigma(t)t^{-1}; t \in T\}$ is a maximal torus of G_σ and $x_\mathcal{O} : \mathbf{C} \rightarrow G_\sigma$, $y_\mathcal{O} : \mathbf{C} \rightarrow G_\sigma$ satisfy $x_\mathcal{O}(\alpha_\mathcal{O}(t_1)) = t_1 x_\mathcal{O}(1) t_1^{-1}$, $y_\mathcal{O}(\alpha_\mathcal{O}(t_1)^{-1}) = t_1 y_\mathcal{O}(1) t_1^{-1}$ for all $t_1 \in T_\sigma$. (We have ${}^\sigma X = \text{Hom}(T_\sigma, \mathbf{C}^*)$ canonically.)

Note that G_σ is simply connected and that $G_\sigma \cong {}^L((({}^L G)^\sigma)^0)$ where ${}^L(\cdot)$ denotes the Langlands dual group and $({}^L G)^\sigma$ denotes the fixed point set of the automorphism of ${}^L G$ induced by σ . Now G_σ is only isogenous to ${}^L(G^\sigma)$ where G^σ is the fixed point set of $\sigma : G \rightarrow G$.

Let $\lambda \in {}^\sigma X^+$. We can view λ both as a character of T and as a character of T_σ . Let V (resp. V') be a finite dimensional complex irreducible representation of G (resp. G_σ) with a non-zero vector η (resp. η') such that $x_i(a)\eta = 0$ for all $i \in I, a \in \mathbf{C}$ (resp. $x_\mathcal{O}(a)\eta' = 0$ for all $\mathcal{O} \in I_\sigma, a \in \mathbf{C}$) and $t\eta = \lambda(t)\eta$ for all $t \in T$ (resp. $t'\eta' = \lambda(t')\eta'$ for all $t' \in T_\sigma$). Now V can be regarded as a representation of $G\langle\sigma\rangle$ whose restriction to G is as above and on which the action of σ satisfies $\sigma(\eta) = \eta$.

Let $\mu \in X$. Let $V_\mu = \{x \in V; tx = \mu(t)x \quad \forall t \in T\}$. Note that $\sigma : V \rightarrow V$ permutes the weight spaces V_μ among themselves. A weight space V_μ is σ -stable if and only if $\mu \in {}^\sigma X$; in this case μ can be viewed as a character of T_σ and we set $V'_\mu = \{x' \in V'; t'x' = \mu(t')x' \forall t' \in T_\sigma\}$.

Theorem (Jantzen [Ja, Satz 9]). *For $\mu \in {}^\sigma X$ we have $\text{tr}(\sigma : V_\mu \rightarrow V_\mu) = \dim V'_\mu$.*

Corollary. *Let $\varpi : T \rightarrow T_\sigma$ be the canonical homomorphism. For any $t \in T$ we have $\text{tr}(t\sigma : V \rightarrow V) = \text{tr}(\varpi(t), V')$.*

The corollary describes completely the character of V on $G\sigma$ in terms of the character of V' since any semisimple element in $G\sigma$ is G -conjugate to an element of the form $t\sigma$ with $t \in T$. Note also that there is a well defined bijection between the set of semisimple G -conjugacy classes in $G\sigma$ and the set of semisimple G_σ -conjugacy classes in G_σ which for any $t \in T$ maps the G -conjugacy class of $t\sigma$ to the G_σ -conjugacy class of $\varpi(t)$; see [L2, 6.26], [Mo].

We now show (assuming that G is not of type A_{2n}) how Jantzen's theorem can be deduced from properties of canonical bases in [L1]. According to [L1], V has a canonical basis B_λ and V' has a canonical basis B'_λ . Also, B_λ (resp. B'_λ) can be naturally viewed as a subset of \mathbf{B} (resp. \mathbf{B}'), the canonical basis of the $+$ part of the universal enveloping algebra attached to the root datum (a) (resp. (b)). Now σ acts naturally on \mathbf{B} (preserving the subset B_λ) and [L1, Theorem 14.4.9] provides a canonical bijection between \mathbf{B}' and the fixed point set of σ on

B. (This theorem is applicable since the Cartan datum of (b) is obtained from the Cartan datum of (a) by the general "folding" procedure [L1, 14.1] which applies to any simplylaced Cartan datum of not necessarily finite type together with an admissible automorphism; here we use that G is not of type A_{2n} .) This restricts to a bijection between B'_λ and the fixed point set ${}^\sigma B_\lambda$ of σ on B_λ . Next we note that B_λ (resp. B'_λ) is compatible with the decomposition of V (resp. V') into weight spaces and from the definitions we see that the bijection above carries $B'_\lambda \cap V'_\mu$ bijectively onto ${}^\sigma B_\lambda \cap V_\mu$. Since $B_\lambda \cap V_\mu$ is a basis of V_μ which is σ -stable we have $\text{tr}(\sigma : V_\mu \rightarrow V_\mu) = \sharp({}^\sigma B_\lambda \cap V_\mu)$. Using the bijection above this equals $\sharp(B'_\lambda \cap V'_\mu)$ and this is equal to $\dim V'_\mu$ since $B'_\lambda \cap V'_\mu$ is a basis of V'_μ . This gives the desired result.

We refer the reader to [FSS, FRS, NS, N1, N2, We] for other approaches to Jantzen's theorem. We thank S. Naito and the referee for pointing out these references to us.

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