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ONE-DIMENSIONAL STOCHASTIC CELLULAR AUTOMATA

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ABSTRACT. We define and discuss the dynamics of cellular automata with the property that at each site in the lattice, or at each coordinate of a point in the domain of the map, a random choice among finitely many local rules is made. We call these stochastic CA's and we set up a mathematical framework for them and provide simple examples in one dimension. They also arise in mathematical models of the spread of viruses where the fast mutation of the virus leads to the appearance of a random choice of local rule. Others forms of stochastic CA's have been studied since the mid-1980's in physics. We conclude with a one-dimensional version of the virus model.

1. Introduction

Cellular automata were introduced in the 1950's by Von Neumann and Ulam to model massively parallel systems, each operating independently under the same local rule. These dynamical systems have many applications in science and mathematics and many studies have been done to understand and classify their longterm behavior. An overview of the literature can be found in

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several places, including [1], [8], or a general text on symbolic dynamics such as [10] or [12]. A large study of properties of cellular automata can be found in [14] and for the various connections to physics a thorough source is [7].

The motivation for this paper is a mathematical model, using a two-dimensional cellular automata, of the spread of the HIV virus in a lymph node that was studied in [3]. Using the definition that a d-dimensional cellular automaton (CA) over a lattice $\mathbb{Z}^d$ is a shift commuting map with finitely many values possible at each lattice site only depending on the value at the site and at a few nearest neighbors (a bounded finite number independent of the site), it turns out that the underlying map in [3] is not a CA. The model used involves random choices at each site so we call it a stochastic CA; the precise definition is given in Section 2. The stochastic CA preserves the salient features of independence of the value from site to site up to local dependence on the state of nearest neighbors. However there are several CA's to choose from due to the underlying unpredictability arising from physiology of the spread of a virus, so a random selection of local rule is made at each site and at each time increment. This aspect of the math simulation captures the random appearance of mutations in a virus, and most importantly is successful in simulating the unusual time scale of the HIV virus, namely that after an initial acute infection the virus appears to be latent and inactive in the system for some time before reappearing on a larger scale.

The details of the mathematical and biological setup in [3] formed the basis for the master's project of J. Hubbs at the University of North Carolina at Chapel Hill under the supervision of the first author [6], where it was observed that the math process used is not a true CA, and therefore understanding the dynamics of the model raised many interesting mathematical questions. In this paper we set aside the application and give a mathematical framework for the model in dimension one, following the analysis with a low-dimensional version of the model.

We define and study the following dynamical system which we refer to as a stochastic cellular automata. Suppose we are given n cellular automata each with radius at most r (the nearest neighbor rules depend only on the r closest sites in each direction);

at each time increment and at each site in the lattice we "roll an n-sided die" to see which local rule to apply. In this paper we provide a preliminary study of the definition and properties of a stochastic CA in the one-dimensional setting only; it was shown in the work of E. Gamber [4] that dynamical properties of CA's vary according to the dimension of the underlying lattice and we wish to isolate the simplest case first to set up the mathematical framework. Stochastic and probabilistic cellular automata have been studied since the mid-80's in various forms and some early papers on this are [2] and [9]; a more recent treatise appears in ([7], Chapter 7). Our study differs from the ones mentioned here in the sense that we take a topological dynamical approach instead of a probabilistic one.

In Section 2 we give the definitions and results supporting the definitions. In Section 3 we give examples and tie them in to some existing classifications of CA's. In Section 4 we extend several dynamical definitions to this setting. Finally we give a one-dimensional version of the model of the virus spread in Section 5.

2. The definition of a stochastic cellular automaton in one dimension

There are several compact spaces to consider for this model. For a finite state space alphabet $A = \{0, 1, \ldots, d-1\}$ we define $X = A^{\mathbb{Z}}$ and we denote the left shift on X by σ_X. A **one-dimensional cellular automaton (CA)** is a continuous map F on X such that $F \circ \sigma_X = \sigma_X \circ F$. For each $x \in X$ and $i \in \mathbb{Z}$, by x_i or $[x]_i$ we denote the i^{th} coordinate of x, and by $x_{\{k,l\}}$, $k < l$ we denote the block of coordinates from x_k to x_l; i.e., $x_{\{k,l\}} \in A^{l-k+1}$.

We define the metric on X to be $d_X(x, v) = \frac{1}{2^k}$ where $k = \min\{|i| \mid x_i \neq v_i\}$; X is compact with respect to the metric topology. By work of Curtis, Hedlund, and Lyndon in the late 1960's [5] the following result allows us to characterize CA's by a local rule. The proof of this result follows from the definition of continuity in the metric topology on X.

Theorem 2.1. [5] *The map F on X is a CA if and only if there exists an integer $r \geq 0$ and a local rule $f : A^{2r+1} \to A$ such that for every $x \in A^{\mathbb{Z}}$,*

$$F(x)_i = f(x_{\{i-r,i+r\}}).$$

By using a larger alphabet if necessary and conjugating, we can assume that $r \leq 1$. However to allow for clarity in the general case we will use r.

We now consider J to be a finite index set (alphabet) with $|J| = n$, and let $\Omega = J^{\mathbb{N}\cup\{0\}}$. At each site in our integer lattice we choose randomly from among n different local rules for n cellular automata indexed by J. The random selection is modeled by the space (Ω, s), the one sided shift space with metric $d_\Omega(\omega, \zeta) = \frac{1}{2^k}$ where $k = \min\{i \mid \omega_i \neq \zeta_i\}$. For each $\omega \in \Omega$, we have the usual shift map

$$[s(\omega)]_j = \omega_{j+1}. \tag{2.1}$$

We now turn to the construction of a stochastic CA. Suppose we have n CA's $F_1, \ldots, F_n$, on $X = A^{\mathbb{Z}}$ with associated local rules $f_1, \ldots, f_n$ respectively and assume each $F_j : X \to X$ has radius at most r.

We define **the stochastic cellular automaton**, or **stochastic CA generated by** $F_1, F_2, \ldots, F_n$ on X as follows. On the space $\Omega \times A^{2r+1}$ we define a local rule by: for each $x \in X$,

$$g(\omega, x_{\{-r,r\}}) = \pi_A(s(\omega), f_{\omega_0}(x_{\{-r,r\}}) = f_{\omega_0}(x_{\{-r,r\}}), \tag{2.2}$$

where π_A denotes projection onto the second coordinate (which is in A).

By definition

$$g : \Omega \times A^{2r+1} \to A;$$

note that g really only depends on the 0^{th} coordinate of ω but this definition will become clear below.

We now look at an infinite product of the spaces Ω, by setting for each i, $\Omega_i = \Omega$, and defining $\overline{\Omega} = \prod_{i=-\infty}^{+\infty} \Omega_i$. We denote each coordinate of a point by $[\overline{\omega}]_i = \omega^{(i)}$, noting that each $\omega^{(i)} = \{\omega_j^{(i)}\}_{j\in\mathbb{N}\cup\{0\}}$ is itself a one-sided sequence from Ω. On $\overline{\Omega}$ we have several shift maps, but we will focus on the one that shifts within each sequence to perform the shift map of (2.1) coordinatewise; it is denoted $\overline{s}$ and defined by:

$$[\overline{s}\,(\overline{\omega})]_j^{(i)} = \omega_{j+1}^{(i)} = [s(\omega^{(i)})]_j. \tag{2.3}$$

Finally we define a random choice of local rule for x given an infinite roll of the die by

$$[F_{\omega^{(i)}}(x)]_i = g(\omega_0^{(i)}, x_{\{i-r,i+r\}}) = f_{\omega_0^{(i)}}(x_{\{i-r,i+r\}}). \tag{2.4}$$

At each coordinate of x we "roll the die" (given by $\omega^{(i)}$) to see which of the n local rules we apply, and we make an independent roll of the die at each coordinate, hence the dependence of F on the coordinate $\omega^{(i)} \in \Omega$. For each infinite collection of independent die rolls given by a point $\overline{\omega} \in \overline{\Omega}$, We write

$$F_{\overline{\omega}} : X \to X$$

to denote the global stochastic CA and its dependence on $\overline{\omega}$. With this definition $F_{\overline{\omega}}$ is not a true CA in the sense of Hedlund, but it is closely related to one as we show here. We reframe this definition in terms of a single transformation.

We define a metric on the product space, $\Omega \times A$ to be the maximum of the two coordinate metrics,

$$\delta((\omega, a), ((\zeta, b)) = \max\{d_\Omega(\omega, \zeta), d_A(a, b)\},$$

where d_A denotes the discrete metric on A.

We then put the product structure on the space $Y = (\Omega \times A)^{\mathbb{Z}}$ in the obvious way. A point in Y has coordinates $y_j = (\omega^{(j)}, x_j)$, for each $j \in \mathbb{Z}$, with $\omega^{(j)} \in \Omega$, $x_j \in A$. The metric ρ on Y is given by:

$$\rho(y, z) = \sum_{j=-\infty}^{+\infty} 2^{-|j|}\delta(y_j, z_j) \tag{2.5}$$

We remark that two points $y, z \in Y$ are close in the metric ρ if and only if the coordinates $y_j = (\omega^{(j)}, x_j)$ are close to the coordinates $z_j = (\zeta^{(j)}, v_j)$ on some central block, say for all $-k \leq j \leq k$. This in turn means that $x_j = v_j$ for j between $-k$ and k, and that $\omega_p^{(j)} = \zeta_p^{(j)}$ for j between $-m$ and m and for $p = 0, 1, \ldots t_j$.

The shift map σ_Y on Y is defined in the obvious way with $[\sigma_Y(y)]_j = y_{j+1} = (\omega^{(j+1)}, x_{j+1})$. The following proposition describes a continuous shift commuting map on a symbol space which represents the stochastic CA, but the presence of the independent randomness introduces an infinite component, preventing it from being a true cellular automaton.

Proposition 2.2. *With the notation above, the map:*

$$\overline{F} : (\Omega \times A)^{\mathbb{Z}} \to (\Omega \times A)^{\mathbb{Z}}$$

defined using local rule:

$$\overline{g} : (\Omega \times A)^{2r+1} \to \Omega \times A,$$

$$\overline{g}(\omega^{\{(-r),(r)\}}, x_{\{-r,r\}}) = (s(\omega^{(0)}), f_{\omega_0^{(0)}}(x_{\{-r,r\}})); \tag{2.6}$$

so that

$$\overline{F}(y)_i = \overline{g}(\omega^{(i)}, x_{\{i-r,i+r\}}) = (s(\omega^{(i)}), f_{\omega_0^{(i)}}(x_{\{i-r,i+r\}}) \tag{2.7}$$

is a continuous shift commuting map of Y.

Moreover, for each $x \in X$,

$$[F_{\overline{\omega}}(x)]_i = \pi_{A_i} \circ \overline{F}(y), \tag{2.8}$$

where π_{A_i} denotes projection onto the i^{th} factor of $X = \prod_{i=-\infty}^{+\infty} A_i$.

Proof. We need to show that $\sigma_Y \circ \overline{F}(y) = \overline{F} \circ \sigma_Y(y)$.

$$\begin{aligned} [\overline{F} \circ \sigma_Y(y)]_i &= \overline{g}(\omega^{(i+1)}, x_{\{i+1-r,i+1+r\}}) \\ &= (s(\omega^{(i+1)}), f_{\omega_0^{(i+1)}}(x_{\{i+1-r,i+1+r\}}), \end{aligned}$$

which is the same as

$$\begin{aligned} [\sigma_Y \circ \overline{F}(y)]_i = [\overline{F}(y)]_{i+1} &= \overline{g}(\omega^{(i+1)}, x_{\{i+1-r,i+1+r\}}) \\ &= (s(\omega^{(i+1)}), f_{\omega_0^{(i+1)}}(x_{\{i+1-r,i+1+r\}}). \end{aligned}$$

The continuity follows from the definition of the metrics and the definition by the local rule given in (2.6).

Since the second coordinate of (2.7) is the same as (2.4) using $i+1$ for i, the last statement follows. □

We see that the local rule $\overline{g}$ is closely related to the local rule for g defined by Equation (2.2). We define higher iterates of $F_{\overline{\omega}}$ as follows:

$$F_{\overline{\omega}}^n(x) \equiv F_{\overline{s}^{n-1}\overline{\omega}} \circ \cdots \circ F_{\overline{s}\overline{\omega}} \circ F_{\overline{\omega}}(x) \tag{2.9}$$

By a slight abuse of notation, we may express the i^{th} coordinate of (2.9) as:

$$[F_{\overline{\omega}}^n(x)]_i = f_{\omega_{n-1}^{(i)}}(f_{\omega_{n-2}^{(i)}} \circ \cdots \circ f_{\omega_0^{(i)}}(x_{\{i-r,i+r\}})_{\{i-r,i+r\}} \cdots)_{\{i-r,i+r\}}.$$

The following summarizes the connection between the stochastic CA $F_{\overline{\omega}}$ and the single transformation $\overline{F}$.

Proposition 2.3. *For every $x \in X$, $x = \{x_j\}_{j\in\mathbb{Z}}$, $\overline{\omega} = \{\omega^{(j)}\}_{j\in\mathbb{Z}} \in \overline{\Omega}$, and $y \in (\Omega \times A)^{\mathbb{Z}}$ with $y_j = (\omega^{(j)}, x_j)$,*

(1) $F^n_{\overline{\omega}}(x) = \left\{\pi_{A_j}\overline{F}^n(y)\right\}_{j\in\mathbb{Z}}$;

(2) *For all $n, m \in \mathbb{N}$, $F^{n+m}_{\overline{\omega}}(x) = F^n_{\overline{s}^m\overline{\omega}} \circ F^m_{\overline{\omega}}(x)$.*

Proof. The first statement follows from Proposition 2.2, (2.8) and the second statement is a direct consequence of (2.9).

□

3. The first examples

We illustrate the definitions above with several examples. In each example all CA's act on the space X=$\{0,1\}^{\mathbb{Z}}$ and each F_i has radius $r \le 1$. The space $\Omega = \{0,1\}^{\mathbb{N}\cup\{0\}}$ so $J = \{0,1\}$. In particular, if $\overline{\omega} \in \overline{\Omega}$, then we have

$$\begin{aligned}\overline{\omega} =& \quad \cdots\omega^{(-2)}\omega^{(-1)}\omega^{(0)}\omega^{(1)}\omega^{(2)}\omega^{(3)}\cdots \\ =& \left\{\begin{array}{cccccccc} \cdots & \omega_0^{(-2)} & \omega_0^{(-1)} & \omega_0^{(0)} & \omega_0^{(1)} & \omega_0^{(2)} & \omega_0^{(3)} & \cdots \\ \cdots & \omega_1^{(-2)} & \omega_1^{(-1)} & \omega_1^{(0)} & \omega_1^{(1)} & \omega_1^{(2)} & \omega_1^{(3)} & \cdots \\ \cdots & \omega_2^{(-2)} & \omega_2^{(-1)} & \omega_2^{(0)} & \omega_2^{(1)} & \omega_2^{(2)} & \omega_2^{(3)} & \cdots \\ & \vdots & & \vdots & & & \vdots & \ddots \end{array}\right\},\end{aligned}$$

where each entry of the array is a 0 or 1.
We consider the following cellular automata on the space X; the tables in the examples list the outcome in row 2 of the local rule applied to the symbol in the center of the triple in row 1 above it, if its nearest neighbors are the ones given.

Example 3.1. (1) (Sum-Product) Let $P : X \to X$ be a cellular automaton for X with radius 1 defined by the local rule

$$p\left(x_{\{i-1,i+1\}}\right) = (x_i + x_{i-1}x_{i+1}) \bmod 2.$$

000	001	010	011	100	101	110	111
0	0	1	1	0	1	1	0

This CA is equicontinuous ([11]); see the definition below.

(2) (Majority) Let $M : X \to X$ be a CA for X with radius 1defined by the local rule

$$m\left(x_{\{i-1,i+1\}}\right) = \left\lfloor \frac{(x_{i-1} + x_i + x_{i+1})}{2} \right\rfloor$$

000	001	010	100	011	101	110	111
0	0	0	0	1	1	1	1

The majority CA is almost equicontinuous [11].

(3) (Identity) $I : X \to X$ where $I(x) = x$ is an equicontinuous CA with radius 0.

(4) (Flip) $L : X \to X$ where $L(x) = 1 - x = (x + 1) \bmod 2$ is also an equicontinuous CA with radius 0.

(5) (Sum) $S : X \to X$ defined by the local rule

$$p\left(x_{\{i-1,i+1\}}\right) = (x_{i-1} + x_{i+1}) \bmod 2.$$

000	001	010	011	100	101	110	111
0	1	0	1	1	0	1	0

has radius 1 and can be shown to be conjugate to the full 2-shift.

(6) (Shift) $\sigma_X : X \to X$ is the left shift on X. Note that this is also a CA with radius 1.

(7) (Constant 1) $C : X \to X$ maps to the constant value 1. It has radius 0 and its local rule is given by

$$c(x_i) = 1.$$

(8) (Right shift) $\sigma_X^{-1} : X \to X$ is the right shift on X.

We show several examples illustrating the behavior of $F_{\overline{\omega}}^n(x)$ as $n \to \infty$ for random choices of x and $\overline{\omega}$ and for various pairings of the examples above. An initial analysis can be done using Wolfram's classes of deterministic one-dimensional CA's which was published in 1984 in [13]. He describes four classes which we characterize loosely here.

Wolfram's Classification. A one-dimensional CA is of Class:

I: if most or all initial choices of x and $\overline{\omega}$ lead to a constant final state;

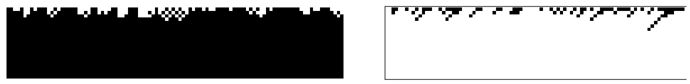

Figure 1. Two Class I stochastic CA's

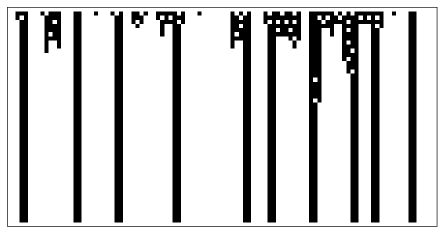

Figure 2. A Class II stochastic CA

II: if final states are extremely simple patterns for most initial states and most $\overline{\omega}$'s;

III: if there appears to be a random or chaotic pattern-free structure in large enough iterates even though some small structures might appear;

IV: if simple structures and randomness appear together, causing the longterm behavior to be the most complex of the four classes.

We interpret our examples using these criteria, and we make more precise remarks about them after presenting definitions in Section 4. In all the figures shown, black denotes 1, and white denotes 0. The top row in each figure is the initial x, the second row, $F_{\overline{\omega}}(x)$, the third $F^2_{\overline{\omega}}(x)$, and so on. Each example is denoted by an ordered pair (F_0, F_1) using the labels from Example 3.1 and each random choice between them is made by tossing a fair coin.

Example 3.2. In Figure 1 we see the result of randomly choosing between the majority CA M and the constant C on the left and the shift σ_X and the constant $C = 0$. In each example we could make choices of $\overline{\omega}$ and x for which the long term behavior would not be constant, but those choices of $\overline{\omega}$ occur with probability 0.

Example 3.3. [Class II] (P, M) We see the evidence of simple longterm but nonconstant behavior in Figure 2.

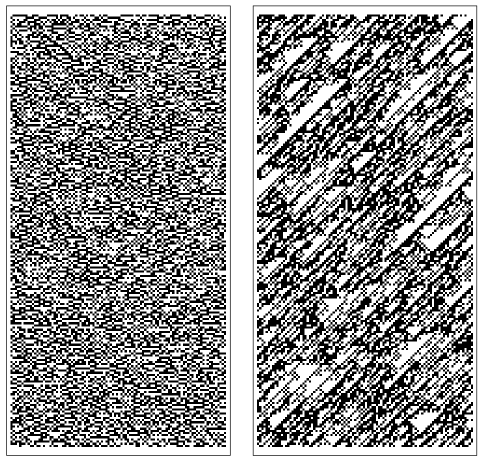

FIGURE 3. Class III CA's (L, S) and (S, σ_X)

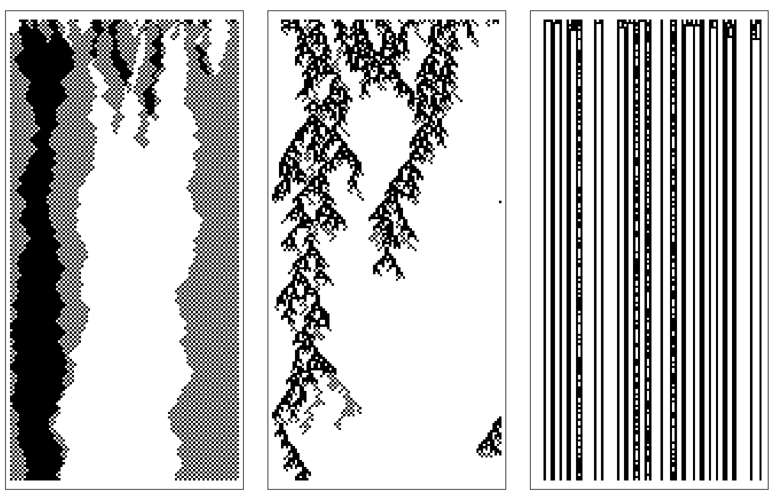

FIGURE 4. Class IV CA's (σ, σ_X^{-1}) and (M, S) and (P, I)

Example 3.4 (Class III). We show (L, S) and (S, σ_X); we notice persistent chaotic features of the CA's in Figure 3 with some small scale structure appearing.

Example 3.5 (Class IV). There are possible Class IV examples as can be seen in Figure 4. The first example on the left is result of making a random choice at each site of shifting to the left or the right, resulting in complex patterns. The middle example is formed by randomly selecting between the majority CA M and the sum S. The example on the right appears to be of Class II except that the sequences trapped in the narrow columns exhibit random patterns.

4. Sensitive dependence on initial conditions and equicontinuity

We now study some topological dynamical properties in the setting of stochastic CA's. Using all the definitions and notation given earlier, we consider n CA's $F_1, \ldots, F_n$, with associated local rules $f_1, \ldots, f_n$ respectively and suppose each $F_j : X \to X$ has radius at most r and that $X = A^{\mathbb{Z}}$. We define the stochastic CA generated by $F_1, F_2, \ldots, F_n$ on X, and denoted $F_{\overline{\omega}}$, as above, using the associated map $\overline{F}$ defined in Proposition 2.2.

In the following definitions $y, z \in (\Omega \times A)^{\mathbb{Z}}$ are given by, $y_j = \left(\omega^{(j)}, x_j\right)$ and $z_j = \left(\zeta^{(j)}, v_j\right)$.

Definition 4.1. *Sensitive Dependence on Initial Conditions:*

(1) We say the map $\overline{F}$ has **sensitive dependence on initial conditions** if $\exists\, \varepsilon = 2^{-k} > 0$ such that $\forall y \in (\Omega \times A)^{\mathbb{Z}}$ and $\forall \delta > 0$, there exist $z \in (\Omega \times A)^{\mathbb{Z}}$ and $n \geq 0$ such that

$$\rho\,(y, z) < \delta \text{ and } d_X\left(\prod_{i=-\infty}^{\infty} (F^n_{\overline{\omega}}(x))_i,\ \prod_{i=-\infty}^{\infty} (F^n_{\overline{\zeta}}(v))_i\right) \geq \varepsilon.$$

(2) We say the stochastic CA $F_{\overline{\omega}}$ has **sensitive dependence on initial conditions** if $\overline{F}$ does. In other words, the sensitive dependence on initial conditions holds for every $\overline{\omega}$.

If $\overline{F}$ ($F_{\overline{\omega}}$) has sensitive dependence on initial conditions we say $\overline{F}$ ($F_{\overline{\omega}}$) is **sensitive**.

Due to the positive expansivity of of the shift on Ω most stochastic CA's exhibit sensitive dependence on initial conditions, including ones coming from CA's of radius 0. Even for some simple examples such as $F_0 = I$ and $F_1 = L$ (flip) as given in Example 3.1, we can show that the resulting stochastic CA has sensitive dependence on initial conditions for $\varepsilon = 1$. We choose and fix any

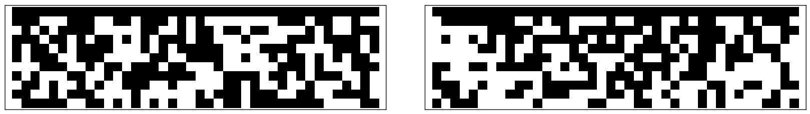

FIGURE 5. The iterates of the point $x = \{\ldots, 1, 1, 1, \ldots\}$ under the CA (I, L)

$x \in X$ and $\overline{\omega} \in \overline{\Omega}$; equivalently we fix $y \in Y$. Given $\delta > 0$ let k be such that $2^{-k} < \delta$. Using the same x we choose $\overline{\zeta}$ so that $\zeta^{(j)} = \omega^{(j)}$ for all $j \neq 0$. Choose $\zeta_i^{(0)} = \omega_i^{(0)}$, $i = 0, 1, \ldots, k-1$, and $\zeta_k^{(0)} \neq \omega_k^{(0)}$. Using z given by $z_j = (\zeta^{(j)}, x_j)$, these choices give rise to the points $y \neq z \in Y$ as above with $\rho(y, z) < \delta$. Using Equation 2.9 we see that

$$[F_{\overline{\omega}}^{j}(x)]_0 = [F_{\overline{\zeta}}^{j}(x)]_0, \ j = 0, \ldots, k,$$

but that

$$[F_{\overline{\omega}}^{k+1}(x)]_0 \neq [F_{\overline{\zeta}}^{k+1}(x)]_0;$$

therefore $d_X \left(\prod_{i=-\infty}^{\infty} (F_{\overline{\omega}}^{n}(x))_i, \prod_{i=-\infty}^{\infty} (F_{\overline{\zeta}}^{n}(v))_i \right) = 1$. Nevertheless this example comes from two equicontinuous CA's (see Definition 4.2 below). This example is illustrated in Figure 5 showing iterates of the point $x = \{\ldots, 1, 1, 1, \ldots\}$ (all 1's).

Sensitive dependence on initial conditions does not always occur though as is seen in the Class I and II examples shown in Figures 1 and 2, and so we turn to the definition of equicontinuity.

Definition 4.2. *Equicontinuity:*

(1) We say $\overline{F}$ **is equicontinuous at** $y \in (\Omega \times A)^{\mathbb{Z}}$ if $\forall \varepsilon = 2^{-k} > 0, \exists \delta > 0$ such that

$$\rho(y, z) < \delta \Longrightarrow d_X \left(\prod_{i=-\infty}^{\infty} (F_{\overline{\omega}}^{n}(x))_i, \prod_{i=-\infty}^{\infty} (F_{\overline{\zeta}}^{n}(v))_i \right) < \varepsilon \ \forall n \geq 0.$$

Note that the inequality for d_X holds if and only if

$$[F_{\overline{\omega}}^{n}(x)]_i = [F_{\overline{\zeta}}^{n}(v)]_i$$

for $|i| \leq k$ and for all n.

(2) For any $\overline{\omega} \in \overline{\Omega}$ with $(\overline{\omega})_j = \omega^{(j)}$, we say the **stochastic CA** $F_{\overline{\omega}}$ **is equicontinous** at $x = \{x_j\}_{j\in\mathbb{Z}}$ if $\overline{F}$ is equicontinuous at $y \in (\Omega \times A)^{\mathbb{Z}}$ with $y_j = \left(\omega^{(j)}, x_j\right)$

(3) We say that $\{F_{\overline{\omega}}\}_{\overline{\omega}\in\overline{\Omega}}$ is **equicontinuous** if for every $\overline{\omega} \in \overline{\Omega}$, $F_{\overline{\omega}}$ is equicontinuous at x for every $x \in X$. This is equivalent to saying $\overline{F}$ is equicontinuous at every $y \in Y$.

The map $\overline{F}$ does not have sensitive dependence on initial conditions if for every $\varepsilon = 2^{-k}$ there exists a $y \in (\Omega \times A)^{\mathbb{Z}}$ and $\exists\, \delta > 0$ such that $\forall z \in (\Omega \times A)^{\mathbb{Z}}$

(4.1)
$$\rho\,(y, z) < \delta \implies d_X\left(\prod_{i=-\infty}^{\infty} (F^n_{\overline{\omega}}(x))_i,\ \prod_{i=-\infty}^{\infty} (F^n_{\overline{\zeta}}(v))_i\right) < \varepsilon\ \forall n \geq 0.$$

This statement has the following coordinatewise interpretation.

Lemma 4.3. *If $\overline{F}$ is not sensitive then for every $p \in \mathbb{N}$ there exist for each $j \in \mathbb{Z}$, an $\omega^{(j)} \in \Omega$ and an $x_j \in A$, and integers $m, K \geq p$ such that $\forall\, v = \{v_j\}_{j\in\mathbb{Z}} \in X$, and $\forall\,\overline{\zeta} = \{\zeta^{(j)}\}_{j\in\mathbb{Z}} \in \overline{\Omega}$, if $v_k = x_k$ for all $|k| \leq m$, and if $\zeta_i^{(k)} = \omega_i^{(k)}$ for all $|k| \leq m$ and $|i| \leq K$, then*

$$F^n_{\overline{\omega}}(x)_{\{-p,p\}} = F^n_{\overline{\zeta}}(v)_{\{-p,p\}}\ \forall n \geq 0.$$

The following extends part of ([11], Theorem 5.1) to our setting.

Proposition 4.4. *If the stochastic CA $F_{\overline{\omega}}$ has a point of equicontinuity then $F_{\overline{\omega}}$ does not have sensitive dependence on initial conditions.*

Proof. The proof follows from the Definition 4.2 and (4.1). □

We want to prove that there are stochastic CA's that have points of equicontinuity despite the instrinsic randomness. We use Kůrka's notion of blocking words to do this.

Definition 4.5. *Blocking Words in $A^{\mathbb{Z}} = X$*

(1) A **word** in a finite alphabet A is a finite sequence of symbols from A; $a = a_0, a_1, a_2, \ldots, a_{n-1}$. The length of a is $|a| = n$; each word $a \in A^{|a|}$.

(2) the word a is a **fixed blocking word** for the CA F if for all $x \in X$,

$$x_{\{0,n-1\}} = a \Rightarrow \forall\, n \geq 0,\; F^n(x)_{\{0,n-1\}} = a. \tag{4.2}$$

A fixed blocking word then is a sequence of symbols which does not change under the CA F. The following is a natural way to extend ([11],Theorem 5.1 (2) $\Rightarrow$ (3)) to the stochastic setting.

Theorem 4.6. *If a is a fixed blocking word for each CA $F_j, j = 1, \cdots, n$ then for the stochastic CA generated by the F_j's there is a point $x \in X$ of equicontinuity and therefore for each $\overline{\omega} \in \overline{\Omega}$, $F_{\overline{\omega}}$ does not have sensitive dependence on initial conditions.*

Proof. Let a be a fixed blocking word for each F_j. We then construct the point x by repeating the block a, i.e., for each $i = dn$, $d \in \mathbb{N}$, $x_{\{i,i+n-1\}} = a$. Then $F^n(x) = x$ for all n. Let $\overline{\omega} \in \overline{\Omega}$ be arbitrary. We claim that the corresponding $y \in Y$ is a point of equicontinuity. Given $\varepsilon > 0$, find k such that $\varepsilon < 2^{-k}$. Write $k = dn + r$, with d, r nonnegative integers. Choose $\delta = 2^{-(d+1)n}$; then for all $y, z \in Y$ with $\rho(y, z) < \delta$, $d_X(x, v) < \delta$ so $F^m(x)_{\{-k,k\}} = F^m(v)_{\{-k,k\}}$ and consists of copies of a. □

Corollary 4.7. *There are examples of stochastic CA's satisfying Theorem 4.6.*

Proof. The example (P, M) from Example 3.3 with fixed blocking word 001100 satisfies the hypotheses of Theorem 4.6. This is shown in Figure 2. □

5. A one-dimensional stochastic CA model of viral spread

We give a one-dimensional (and therefore overly simplified) version of the model of the spread of a virus described in [3]. Our physiological explanation here is based on the discussion in [3] but is not necessarily supported by empirical data since that study used a two-dimensional CA. Other CA models of viral spread are discussed in ([7], Chapter 8). Since the mathematical definitions and dynamical properties of stochastic CA's can be generalized to higher dimensions and this example yields similar results in dimension one, we include it here.

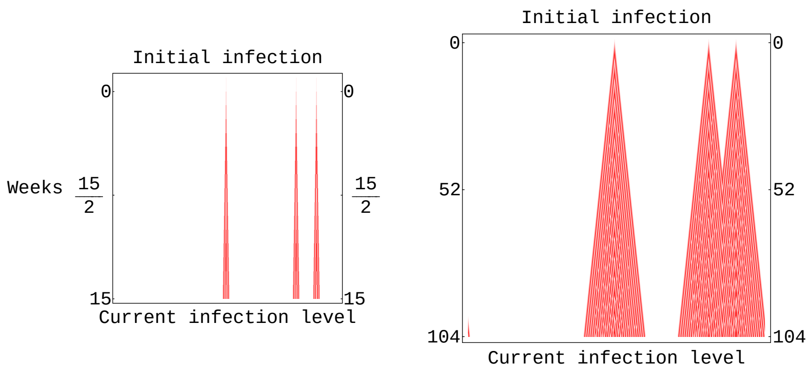

FIGURE 6. The level of infection after 15 weeks (left) and after 2 years (right)

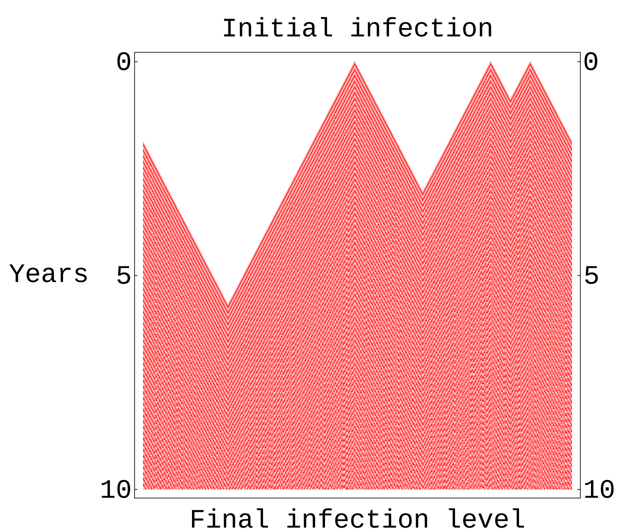

FIGURE 7. The level of infection after ten years

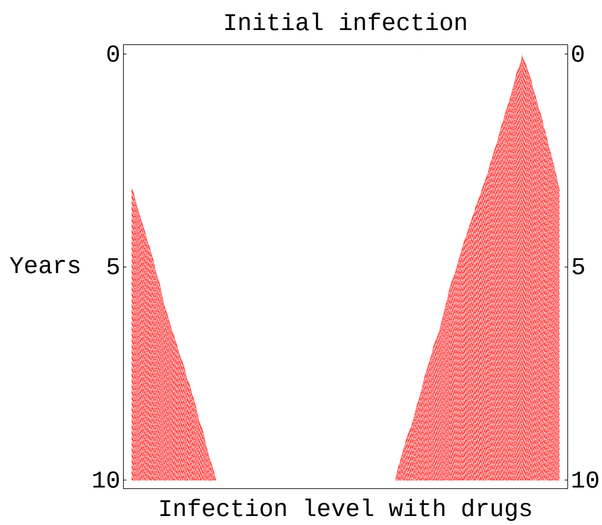

Figure 8. The level of infection after ten years and drug treatment

We assume that we are in a closed circulatory system consisting of approximately 1000 circulating cells or sites. At the onset of the experiment we introduce a sparse randomly distributed number of infected cells into an otherwise healthy array of cells. The initial infection viral load is approximately .2 % so we have around two infected cells in our initial point.

A healthy cell in the circulatory system gets infected if and only if it is touching an infected cell; this occurs in one unit of time (usually a week). An infected cell stays infected for approximately 5 weeks at which time it is depleted (and dies). Usually (98 % of the time in our model) the cell is immediately replenished by a healthy cell due to the activity of the immune system, but occasionally a cell is not replaced or, when replaced it is instantaneously infected, since the virus is still present. This gives us the set of local rules for our stochastic CA model.

Each cell is in one of 7 states: healthy (state 0), infected (states $1, \ldots 5$), and dead (state 6). The explanation for the levels of infected cells is that there is a time lag between the time of infection and the total depletion or death of a cell. We give the local rules for the two different CA's in the stochastic CA; for the CA F_0, which occurs 98 % of the time, a depleted cell is replaced by a healthy cell (a 6 becomes a 0). For the CA F_1, a depleted cell is replaced by an infected cell (a 6 becomes a 1).

The CA F_0:

000	$x_{i-1}\,6\,x_{i+1}$	all other $x_{i-1}\,x_i\,x_{i+1}$
0	0	$x_i + 1 \mod 7$

The CA F_1:

000	$x_{i-1}\,6\,x_{i+1}$	all other $x_{i-1}\,x_i\,x_{i+1}$
0	1	$x_i + 1 \mod 7$

The CA F_0 is chosen 98 % of the time by an "unfair" die. In Figures 6 and 7 the healthy (uninfected) cells are white. The infected cells change color slowly and a dead cell is deep red. We notice that the time scale reflects some empirical data on the spread of the HIV virus, namely the slow but steady spread of the virus even when after several years the level of the virus still appears to be quite low. The appearance of most runs of this experiment is a chaotic mix of cells, most of them not healthy, by the time one reaches the 10 year mark. Every run of this experiment on a computer shows it to be a Class III example. By applying some of the techniques from the previous section and moving the model into two and three dimensions we hope to understand this and other applications in more detail. Also it is desirable to understand how an introduction of drugs can change the local rules in such a way as to change the stochastic CA into a Class I or II example. As a simple illustration we could define the additional local rule which simulates the effect of introducing virus destroying drugs into the system; a simple assumption might be that the drugs will kill off viruses infected for two weeks or less (before they have had time to mutate into something unaffected by the particular drug), unless the cell touches a more infected cell so that a new rule is given as follows.

The CA F_2:

$x_{i-1} + x_i + x_{i+1} < 3$	all other $x_{i-1}\,x_i\,x_{i+1}$
0	$x_i + 1 \mod 7$

Allowing the rules F_0 and F_2 to occur equally often, we see from Figure 8 that after ten years the model shows a significantly different outcome though not necessarily a drop to a Class II process.

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