NL3621 **Ergodic theory**

Basic overview

Ergodic theory is the statistical study of groups of motions of a space, either physical or mathematical, with a measurable structure on it. The origins of ergodic theory can be traced back to the mid-nineteenth century when containers of gas particles were first viewed as sets of randomly moving objects rather than as a collection of individual particles moving under known forces. The word ergodic was introduced by Ludwig Boltzman in the context of the statistical mechanics of gas particles, and it comes from two Greek words "ergon" (work) and "odos" (path).

The mathematical setting in which ergodic theory is studied is as follows. Starting with a space X that represents all possible states of some system which changes over time under known forces, a point $x \in X$ corresponds to one state in the space. The measurable structure consists of a collection of measurable sets $\mathcal{B}$ on X along with a probability measure μ. The measure μ is a function that associates to each set B in $\mathcal{B}$ a number between 0 and 1; this number is the measure of B and we write its measure as $\mu(B)$. A probability measure has the property that $\mu(X) = 1$. Instead of tracking the path of each object in the system, one studies the statistical properties of the motion. The subject evolved from statistical mechanics applied to the study of systems of gas particles moving according to classical laws of physics. In principle the path of each gas particle can be tracked and its entire history and future can be known; in practice the complete determination of the paths of the gas molecules is not feasible.

After t units of time, $F_t(x)$ is the point in X corresponding to where x ends up. If $t \in \mathbb{R}$, then the *orbit* of x is the set $\{F_t(x)|t \in \mathbb{R}\}$; F_t is called a flow and defines an action of the group $\mathbb{R}$ on X. Frequently one uses discrete time intervals and writes $T^n(x) \equiv F_n(x)$ for each integer n, so the orbit of x is a discrete set $\{T^n(x)|n \in \mathbb{Z}\}$ and the group acting on X is $\mathbb{Z}$. In the discrete setting the transformation T is the generating map T^1, and T^n is T composed with itself n times. In classical ergodic theory the measure μ is preserved under the action; i.e., for any set $A \in \mathcal{B}$, $\mu(A) = \mu(F_t A)$ for all $t \in \mathbb{R}$ or $\mu(A) = \mu(T^n A)$ for all $n \in \mathbb{Z}$. One of the main advantages of the ergodic theoretic point of view is that one can ignore some orbits if they only form a set of measure 0 in the space X; therefore one uses the terminology μ - *almost everywhere* (or *for* $\mu - a.e.\ x$) to refer to a property that holds on a set of points of measure 1 in X but perhaps fails to hold on some set of measure 0.

Boltzmann's original ergodic hypothesis has come to be known as the statement that *time average equals space average*; he conjectured (in a certain classical setting) that for any integrable function f, for $\mu - a.e.\ x$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int_X f d\mu. \tag{1}$$

In this expression, the time average of f for n time steps along the orbit of x is represented by averaging $f(x)$ with $f(Tx)$, $f(T^2x)$, $\ldots$, and $f(T^{n-1}x)$, since each application of T represents the passage of one unit of time (the left hand side of the expression). The space average of f is obtained by integrating f over the entire space X, giving the right hand side of the expression. For a flow the ergodic hypothesis is

$$\lim_{t\to\infty}\int_0^t f(F_t x)dx = \int_X f d\mu. \tag{2}$$

The conjecture is false as stated; it only holds when the action is μ-ergodic, and this realization led to the definition of an ergodic action.

The setting of ergodic theory has been greatly enlarged to include the actions of many other groups (Zimmer, 1981). A nonsingular action Φ of a group G on a space $(X, \mathcal{B}, \mu)$ consists of an action of G on X such that the map $\Phi : G\times X \to X$ is measurable and for each $g \in G$ the map $\phi_g(x) = \Phi(g, x)$ is a nonsingular automorphism of X, i.e., ϕ_g is an invertible measurable transformation and for any $B \in \mathcal{B}$, $\mu(B) = 0$ if and only if $\mu(\phi_g^{-1}B) = \mu(\phi_{g^{-1}}B) = 0$. The group operation on G is reflected in the action since $\phi_{g_1 g_2}(x) = \phi_{g_2}(\phi_{g_1}x)$ for all $g_1, g_2 \in G$ and almost every (μ–a.e.) x.

An action is *ergodic* if whenever $\phi_g(A) = A$ for all $g \in G$ then $\mu(A) = 0$ or 1. The study has been extended beyond group actions to the study of ergodic equivalence relations, and the assumption that the measure μ be a probability measure is frequently dropped. One also considers the actions of semigroups when the action being studied is not invertible.

The Birkhoff ergodic theorem for a measure-preserving transformation T states that the limit of the left hand side in Equation (1) exists for μ–a.e. x, is an integrable function f^*, and $f^*(Tx) = f^*(x)$ for μ-a.e. x. Furthermore f^* is the constant $\int_X f d\mu$, the space average of f, precisely when the action is ergodic (Birkhoff, 1931). Any point x satisfying the theorem is called a generic point and generates a *typical orbit*. There are many ergodic theorems; perhaps the simplest is the von Neumann ergodic theorem (von Neumann, 1932). By definition, a function $f \in L^2$ if $\int |f|^2 d\mu < \infty$; moreover defining $f \circ T^k(x) = f(T^k x)$, for measure-preserving T, $f \circ T^k \in L^2$ as well. Von Neumann's theorem is also called the Mean or L^2 Ergodic Theorem because while no individual orbit is tracked (no x appears in the statement), the average difference (L^2 integral) between the time average and the limit function f^* must be small.

Theorem 1 *(***Von Neumann or L^2 Ergodic Theorem***) If T is a measure-preserving transformation and $f \in L^2$, then there is a function $f^* \in L^2$ such that*

$$\int_X \left|\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k - f^*\right|^2 d\mu \to 0 \quad as \quad n\to\infty.$$

If $(X, \mathcal{B}, \mu)$ is a probability space and the transformation T (or group action) on X is non-ergodic, there is a disintegration or decomposition of μ into measures μ_y, indexed by points y in another probability space Y, with its own measure ρ, such that $\mu(A) = \int_Y \mu_y(A)d\rho$ for every $A \in \mathcal{B}$. Furthermore the limit function f^* in the ergodic theorems is constant with respect to each measure μ_y; i.e., T is μ_y ergodic for $\rho - a.e.\ y$. The decomposition is independent of the function f; this is referred to as the *ergodic* *decomposition* of the measure μ with respect to T (Rohlin, 1949). One of the central problems in ergodic theory is to determine when two measure-preserving group actions are conjugate via a measure-preserving isomorphism. To this end, invariants such as entropy and spectral properties have been the subject of much study.

There is a hierarchy of statistical properties associated to ergodic actions, including weak mixing, mixing, K-automorphisms, and Bernoulli automorphisms. As a transformation moves up the hierarchy (in the order listed), the more chaotic the behavior of the system is expected to be.

Applications to dynamical systems and chaos

The simplest example of an ergodic transformation is irrational rotation on the circle with respect to Lebesgue measure. The most random transformation is a Bernoulli shift with an independent identically distributed measure on it; this includes coin tosses.

Overlaps of the topological setting of dynamical systems with ergodic theory exist, and much of ergodic theory highlights the interface between the topological and measurable structure of a group G acting on $(X, \mathcal{B}, \mu)$. In 1935 Hedlund proved the ergodicity of the geodesic flow on the unit tangent bundle of a surface of constant negative curvature; in 1940 Hopf extended the result to establish ergodicity of the geodesic flow on arbitrary manifolds with negative sectional curvature. In this setting the invariant measure is the Liouville measure.

Modern ergodic theory was started by Andrei Kolmogorov with the formal development of Boltzmann's notion of entropy, and developed in the 60's and 70's to include many differentiable actions. Applications include fluid dynamics, coding theory, number theory, complex dynamics, and cellular automata.

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See also Chaotic dynamics, Dynamical systems, Entropy, Symbolic dynamics

Further Reading

Birkhoff, G. 1931. Proof of the ergodic theorem. *Proceedings of the National Academy of Sciences U.S.A.*, 17: 656–660

Cornfeld, I., Fomin, S., & Sinai, Y. 1982. *Ergodic Theory*, New York: Springer

Furstenberg, H. 1981. *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton: Princeton University Press

Halmos, P. 1956. *Ergodic Theory*, New York: Chelsea Publishing

Katok, A. & Hasselblatt, B. 1995. *Introduction to the Modern Theory of Dynamical Systems*, Cambridge and New York: Cambridge University Press

Petersen, K. 1983. *Ergodic Theory*, Cambridge and New York: Cambridge University Press

Rohlin, V. A. 1952. On the fundamental ideas of measure theory. *American Mathematical Society Translations*, (1) 10: 1–54 (original Russian article 1949) von Neumann, J. 1932. Proof of the quasiergodic hypothesis. *Proceedings of the National Academy of Sciences U.S.A.*, 18: 70–82

Walters, P. 1982. *An Introduction to Ergodic Theory*, New York: Springer

Zimmer, R. 1984. *Ergodic Theory and Semisimple Groups*, Boston: Birkhäuser