Poisson homotopy algebra
An idiosyncratic survey of homotopy algebraic topics related to Alan's interests

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Dedicated to Alan Weinstein on his 60th birthday

Abstract

Homotopy algebra is playing an increasing role in mathematical physics. Especially in the Hamiltonian and Lagrangian settings, it is intimately related to some of Alan's interests, e.g. Courant and Lie algebroids. There is a comparatively long history of such structure in cohomological physics in terms of equations that hold mod exact terms (typically, divergences) or only 'on shell', meaning modulo the Euler-Lagrange equations of 'motion'; more recently, higher homotopies have come into prominence. Higher homotopies were developed first within algebraic topology and may not yet be commonly available tools for symplectic geometers and mathematical physicists.

This talk is planned as a gentle introduction to the basic point of view with a variety of applications to substantiate its relevance. Most technical details are supplied by references to the original work or to [MSS02].

Contents

1 Introduction

Cohomological physics is a phrase I introduced some time ago in the context of anomalies in gauge theory, but it all began with Gauss' invention of the (elctromagnetic or asteroidal) linking number in 1833. The cohomology referred to in Gauss was that of differential forms, div, grad, curl and especially Stokes Theorem (the de Rham complex).

2 Basic concepts: DG Space of states, maps and homotopies

Whatever 'states' are physically, mathematically it is crucial that they form a vector space, in fact, usually a Hilbert or pre-Hilbert space. In cohomological physics, the *physical* space of states H is the (co)homology of a dg vector space, $V = \oplus V^n, d : V^n \to V^{n+1}$ with $d^2 = 0$. (As is more common in physics, we have adopted the *co*homological conventions with the grading as a superscript and the differential of degree 1. Of course there is a corresponding theory with differentials of degree -1 for which we would indicate the grading with a subscript. These conventions are equivalent just by raising/lowering: $V^n \Leftrightarrow V_{-n}$.) The space H is often considered as a subspace of the dg vector space by some (implicit) choice of representatives. In physical language, this might be referred to as *gauge fixing.*

Although much of physics is phrased in terms of manifolds and even analysis, my point of view is almost entirely (differential graded) algebraic, eg. think of an *algebra of observables* without considering them as functions.

Maps (morphisms) of dg vector spaces $f : (V, d_V) \to (W, d_W)$ are linear maps of degree 0 which respect the differentials: $d_W f = f d_V$. For cochains/differential forms on topological spaces/manfolds, maps of spaces induce morphisms of the dg vector spaces. Assuming the differentials are of degree 1, a homotopy between two such maps is a linear map $h : (V, d_V) \to (W, d_W)$ of degree -1 such that

$$f - g = d_W h + h d_V.$$

Homotopies in the topological or smooth sense induce such dg homotopies. Notice, when applied to cocycles (representatives of physical states), $f = g$ mod exact terms.

Higher homotopies refer to homotopies of homotopies, and so on. For example, for given f and g with two such homotopies h and k as above, a second level homotopy is a linear map $h_2 : (V, d_V) \to (W, d_W)$ of degree -2 such that

$$h - k = d_W h_2 - h_2 d_V.$$

In full generality, higher homotopies refer to a family of linear maps $h_n : (V, d_V) \to (W, d_W)$ of degree $-n$ such that $d_W h_n - (-1)^n h_n d_V$ satisfies some relation among the h_i for $i < n$.

It is time to look at examples.

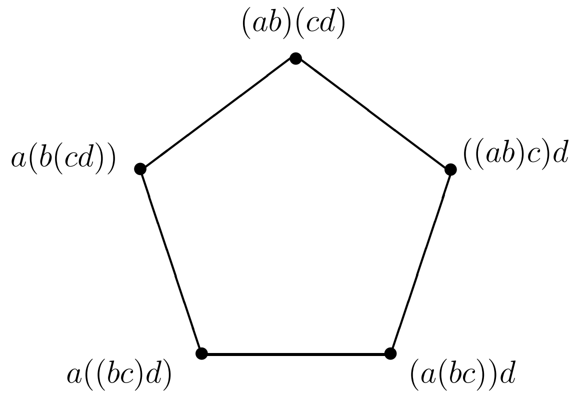

Figure 1: The pentagon K_4

3 A_∞-structure

The examples currently most relevant to physics and Poisson structure are those of A_∞- and L_∞-structures [MSS02]. The topological version of A_∞-structure came first and is the easiest to visualize. It also has an obvious relevance to open string field theory (OSFT) [Kaj03] as L_∞-structure has to closed string field theory (CSFT) [Zwi93].

Consider the space of *based* loops ΩX on a space X with base point $*$. That is, a based loop is a map λ of the unit interval I into X such that $\lambda(0) = * = \lambda(1)$. Because we define the 'product' of two loops by reparameterizing the result of following one loop by another, this product is only homotopy associative.

Consider a specific associating homotopy $h(a, b, c)$ from $a(bc)$ to $(ab)c$. There are 5 ways of parenthesizing the product of 4 loops, which results in a pentagon of loops, where the sides represent a single application of a specific associating homotopy $h(a, b, c)$ from $a(bc)$ to $(ab)c$. For example, the bottom edge from left to right of Figure 1 is given by $h(a, bc, d)$. By looking at the parameterizations in more detail, it can be seen that the pentagon can be filled in by a 2-parameter family of loops.

Now there are 14 ways of parenthesizing the product of 5 loops and so on. The combinatorics, in general for n-loops, can be realized in terms of a polyhedron, called an *associahedron* and denoted K_n, described as a convex polytope with one vertex for each way of associating n ordered variables, that is, ways of inserting parentheses in a meaningful way in a word of n letters. For $n = 5$, a portrait due to Masahico Saito is in Figure 2.

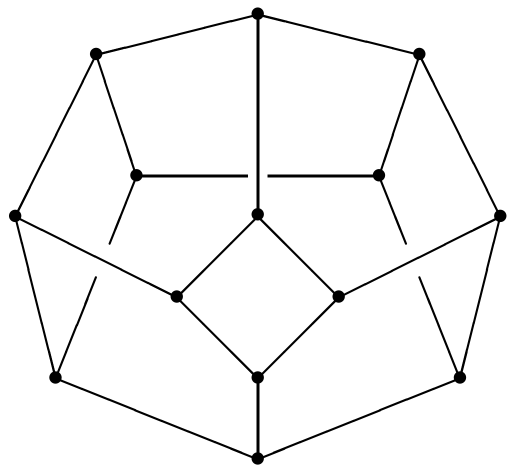

Figure 2: Saito's portrait of K_5.

An A_∞*-space* Y is a topological space Y together with a family of maps

$$m_n : K_n \times Y^n \to Y$$

which fit together in a way suggested by the pentagon.

Now consider a *chain complex functor* from topological spaces to dg modules (over a commutative ground ring) which respects products. Applied to an A_∞-space, such a functor reveals an algebraic structure generalizing that of a dg associative algebra.

Definition 3.1 *An* A_∞-algebra *(or strongly homotopy associative algebra) consists of a graded module* V *with maps*

$$m_n : V^{\otimes n} \to V \text{ of degree } 2-n$$

satisfying suitable compatibility conditions $(A_n)_{n\geq 1}$.

In particular,

(A_1) $m_1 = d$ is a differential,

(A_2) $m = m_2 : V \otimes V \to V$ is a chain map, that is, d is a derivation with respect to $m = m_2$,

(A_3) $m_3 : V^{\otimes 3} \to V$ is a chain homotopy for associativity of the multiplication m, i.e.

$$m_3 d^{\otimes 3} + dm_3 = m(m \otimes 1) - m(1 \otimes m),$$

where $d^{\otimes 3}$ denotes $d \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d$,

(A_4) m_4 is a 'higher homotopy' such that $m_4 d^{\otimes 4} - dm_4$ has five terms, corresponding to the edges of the pentagon K_4:

$$m_4 d^{\otimes 4} - dm_4 = m_3(m_2 \otimes 1 \otimes 1 - 1 \otimes m_2 \otimes 1 + 1 \otimes 1 \otimes m_2) - m_2(m_3 \otimes 1 + 1 \otimes m_3).$$

An alternate formulation generalizes the bar construction on an associative differential graded algebra. Define the suspension sA of a graded vector space A by shifting the grading down: $(sA)^n = A^{n+1}$.

Alternate Definition 3.2 *An* A_∞-algebra *structure on a positively graded vector space* A *is equivalent to a coderivation differential* δ *of degree* 1 *with respect to the total grading on the tensor coalgebra* $\mathbf{T}^c(sA)$ *on the suspension of the graded vector space* A*. As a coderivation,* δ *is determined by the formula* $\delta = \delta_1 + \delta_2 + \cdots$*, where*

$$\delta_n(sa_1 \otimes \cdots \otimes sa_n) := \epsilon \cdot sm_n(a_1 \otimes \cdots \otimes a_n), \text{ for } a_1, \ldots, a_n \in A,$$

and ϵ *is an appropriate sign.*

4 L_∞-structure

Although there was no topological predecessor, the notion of an L_∞-algebra follows a similar pattern algebraically.

Following our example of having associativity satisfied only up to homotopy, we can do the same for the Jacobi identity and then consider higher homotopies. That is, the Jacobiator is not zeo but is exact, i.e. a boundary. One can give the full definition in terms of a family of maps

$$l_n = [-, \cdots, -] : \mathfrak{g}^{\otimes n} \to \mathfrak{g}$$

which are suitably 'skew symmetric' and compatible, but it is definitely a lot simpler to give the definition in terms of $\Lambda s\mathfrak{g}$.

Definition 4.1 *An* L_∞-algebra *structure on a graded vector space* $\mathfrak{g}$ *is equivalent to a coderivation differential* δ *of degree* 1 *with respect to the total grading on the graded symmetric coalgebra* $\Lambda^c(s\mathfrak{g})$ *on the suspension of the graded vector space* $\mathfrak{g}$.

If you'd like some hands-on examples, consider very small finite dimensional L_∞-algebras. There are two versions of the classification, depending on whether we consider L_∞-algebras in the original Z-graded sense or the super, i.e. $= Z/2Z$-graded, sense. There are classifications here by, respectively, Daily [Dai02] or Fialowski-Penkava [FP03, **?**] for very small dimensional examples.

In the Z-graded situation, particularly important are the cases considered below with $\mathfrak{g}^{-1} \to \mathfrak{g}^0$ and $\mathfrak{g}^0 \to \mathfrak{g}^1$. The first of these, but with d of degree -1, i.e. $\mathfrak{g}_1 \to \mathfrak{g}_0$, are considered extensively and categorically as Lie 2-algebras by Baez [BC03].

Now what does this have to do with physics or symplectic geometry or Alan's interests?

The answers include moment maps, symplectic reduction and Courant algebroids.

4.1 Courant algebroids

The most straightforward connection with Alan's interests appears in his paper with Roytenberg [RW98] on Courant algebroids. There is no point in repeating the very clear exposition in their original paper, so I will mention only the salient facts in re: higher homotopies.

Definition 4.2 *A* Courant algebroid *is a vector bundle $E \to M$ equipped with a non-degenerate bilinear form $< \cdot, \cdot >$ on the bundle, a skew symmetric bracket $[\cdot, \cdot]$ on the space of sections $\Gamma(E)$ and a bundle map $a : E \to TM$ satisfying 5 properties.*

Here at the Alanfest (in fact, on the train in from the airport), I learned that Kosmann-Schwarzbac has simplified the defintion, but in the non-skew-symmetric form.

The skew bracket does not in general satisfy the Jacobi identity, but property number 5 addresses the defect in terms of a map $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ related to the deRham differential on M. Roytenberg and Weinstein consider the following small complex:

$$X^{-1} = C^\infty(M) \to X^0 = \Gamma(E) \tag{1}$$

with the differential l_1 given by $\mathcal{D}$. They define a very specific L_∞-structure as follows:

l_2 is given by the bracket on $X_0 \otimes X_0$ and by $< \cdot, \mathcal{D} \cdot >$ on $X_0 \otimes X_1$ and 0 in higher degrees,

l_3 is given by one third of the sum of the cyclic permutations of $< [\cdot, \cdot], \cdot >$ on $X_0 \otimes X_0 \otimes X_0$ and 0 otherwise,

while l_n for $n > 3$ is 0.

Note the similarity to Drinfel'd's conditions.

Of course, many of these 0's follow just from the fact that $X_n = 0$ for $n \geq 2$, in contrast to the L_∞-structures that appear in the work of Fulp, Lada and myself concerning higher spin algebras [FLS02b, FLS02a] (see section 6.1 below).

5 Homological reduction of constrained Poisson algebras

Cohomological physics had a major breakthrough with the 'ghosts' introduced by Fade'ev and Popov [FP67]. These were incorporated into what came to be known as *BRST cohomology* (Becchi-Rouet-Stora [BRS75] and Tytutin [Tyu75]) and which was applied to a variety of problems in mathematical physics. There the ghosts were reinterpreted by Stora [Sto77] and others in terms of the Maurer-Cartan forms in the case of a finite dimensional Lie group and more generally as generators of the Chevalley-Eilenberg cochain complex [CE48] for Lie algebra cohomology.

If, as geometers, you feel more comfortable with manifolds, one can make the following algebra seem more palatable as functions on 'supermanifolds', but most (all?) of the work is just algebraic (homological).

WARNING! The term 'BRST cohomology' has a variety of meanings in the existing literature. From time to time, it threatens to be used for any cohomology in physics, at least if the coboundary operator is called 'Q'. At other times, it refers (only implicitly) to the case in which the Lie algebra is the Virasoro algebra. I prefer to reserve the term for situations in which the coboundary operator has at least some part corresponding to that of Chevalley-Eilenberg.

Such is the case for the ghost technology for the cohomological reduction of constrained Poisson algebras, introduced by Batalin, Fradkin and Vilkovisky [BF83, FF78, FV75], which extended the complex of BRST by adjoining odd generators, called *ghosts* and *anti-ghosts*, thus reinventing the

Koszul-Tate [Tat57] resolution of the ideal of constraints and producing a synergistic combination of both Chevalley-Eilenberg and resolution cohomology. Here it was that I saw the essential features of a strong homotopy Lie algebra (L_∞-algebra).

5.1 Moment maps, momaps and symplectic reduction

The setting is one of Alan's favorites [Wei02], that of a moment map, though generalized in an important way. Alan considers:

> a phase space with a symmetry group consists of a manifold P equipped with a symplectic structure ω and a hamiltonian action of a Lie group G. By the latter, we mean a symplectic action of G on P together with an equivariant moment map J from P to the dual $\mathfrak{g}^*$ of the Lie algebra of G such that, *for each* $v \in \mathfrak{g}$, *the 1-parameter group of transformations of* P *generated by* v *is the flow of the hamiltonian vector field with hamilitonian* $x \to< J(x), v >$*.* The map J is called the *momentum map* (or, by many authors, *moment map*) of the hamiltonian action. If one is simply given a symplectic action of G on P, any map J satisfying the condition in italics above, even if it is not equivariant, is called a momentum map for the action.

By contrast, Batalin-Fradkin-Vilkovisky consider *constraints* on the symplectic manifold P to be primary.

A *Hamiltonian system with constraints* means we have functions $\phi_\alpha : P \to \mathbf{R}, 1 \leq \alpha \leq r$, the constraints. Solutions of the system are constrained to lie in a subspace $V \subset P$ given as the zero set of a smooth *momap* $\phi : P \to W = \mathbf{R}^r$ with components ϕ_α. In contrast to the more restrictive case in which $W = \mathbf{R}^r$ has the structure of a Lie algebra $\mathfrak{g}$ and ϕ is assumed to be equivariant with respect to the action of the corresponding Lie group on P, here we do not assume any Lie group G action. To emphaisze this, I refer to ϕ as a *momap.* (This also avoids the moment versus momentum controversy revealed to me last night.) The algebra $C^\infty(V)$ is given by $C^\infty(P)/I$ where I is the ideal generated by the ϕ_α. Dirac calls the constraints *first class* if I is closed under the Poisson bracket. In terms of the constraints, the condition is then

$$\{\phi_\alpha, \phi_\beta\} = f^\gamma_{\alpha\beta}\phi_\gamma,$$

where we have *structure functions* $f^{\gamma}_{\alpha\beta}$ on P, not structure constants. In other words, we have a Lie algebroid with anchor map $a : C^\infty(P) \to \Gamma(TP)$ given by the Hamiltonian vector field associatied to a function. If we let W denote the *vector space* spanned by the ϕ_α, physicists speak of W as an *open algebra* since the bracket defined on W does not close in W. Compare this with Lie's notion of *function group* [Lie90] as discussed by Alan [Wei02].

In this first class case, the Hamiltonian vector fields X_{ϕ_α} determined by the constraints are tangent to V (where V is smooth) and give a foliation $\mathcal{F}$ of V. Similarly, $C^\infty(P)/I$ is an I-module with respect to the bracket. (In symplectic geometry, the corresponding variety is called *coisotropic*. The passage from P to $V/\mathcal{F}$ is known as *symplectic reduction*.) The true physics of the system is the induced system on the space of leaves $V/\mathcal{F}$. If that space is a smooth manifold, $C^\infty(V/\mathcal{F})$ is the true algebra of observables. When $C^\infty(V/\mathcal{F})$ makes no sense, the Batalin-Fradkin-Vilkovisky construction provides a replacement, as described below (see [HT92] for a comprehensive treatment).

In this context, the classical BRST construction, at least as developed by Batalin-Fradkin-Vilkovisky in the case of regular constraints, is a *homological model* for $C^\infty(V/\mathcal{F})$ or rather for the full de Rham complex $\Omega(V, \mathcal{F})$ consisting of forms on vertical vector fields, those tangent to the leaves.

The model is constructed as follows:

First, consider the most common case of an equivariant moment map $\phi : P \to W = \mathfrak{g}^*$ with respect to a Lie group action of G on P where $\mathfrak{g}$ is the Lie algebra of G. Let A denote $C^\infty(P)$ considered as a Poisson algebra. Extend A as a graded commutative algebra to

$$\mathcal{BFV} = A \otimes E\mathfrak{g}^* \otimes E\mathfrak{g} \tag{2}$$

where E denotes the exterior algebra. Extend the Poisson bracket $\{\cdot, \cdot\}$ (still of degree 0) as determined by the fundamental pairing $\mathfrak{g}^* \otimes \mathfrak{g} \to \mathbf{R}$. Note: Elements of $\mathfrak{g}$ are called anti-ghosts and have degree -1 while elements of $\mathfrak{g}^*$ are called ghosts and have degree 1. Now make $\mathcal{BFV}$ a dg Poisson algebra by defining

$$d_{\mathcal{BFV}} = d_K + \delta^* \tag{3}$$

where δ^* is the Chevalley-Eilenberg coboundary and d_K is the Koszul differential on $A \otimes E\mathfrak{g}$ regarded as a resolution of the ideal of constraints. In terms of a basis $\{e_\alpha\}$ for $\mathfrak{g}$ so that $\phi_\alpha = e_\alpha \circ \phi$, this means that d_K is the graded derivation determined by

$$d_K(e_\alpha) = \phi_\alpha.$$

Because we have a strict Lie group action and, hence, structure constants, it is straightforward to verify $d^2_{\mathcal{BFV}} = 0$, but this is not the case for our momaps. The definition of the algebra is no problem:

$$\mathcal{BFV} = A \otimes EW \otimes EW^* \tag{4}$$

and $d_K + \delta^*$ is defined as before but fails to square to 0, essentially because we now have structure *functions*. In the regular case, the brilliance of Batalin-Fradkin-Vilkovisky was to define $d_{\mathcal{BFV}}$ by adding terms of higher order to $d_K + \delta^*$ so that $(d_{\mathcal{BFV}})^2 = 0$. With hindsight, the existence of such terms of higher order was due to the fact that $A \otimes E\mathfrak{g}$ provided a resolution of the ideal of constraints, thus permitting the techniques of Homological Perturbation Theory [Gug82, GL89, GLS90, GS86, Hue84, HK91]. However, the proof crucially involves keeping $d_{\mathcal{BFV}}$ as an inner derivation $\{Q, \cdot\}$.

The point of doing this is:

Theorem 5.1 *If the first class constraints generate a regular ideal, then the cohomology of $\mathcal{BFV}$ is isomorphic to the cohomology of $\Omega(V, \mathcal{F})$ with respect to the leaf-wise exterior differential. In particular, $H^0(\mathcal{BFV})$ is isomorphic to $H^0(\Omega(V/\mathcal{F}))$, the algebra of 'observables' on the reduced phase space.*

In the more general non-regular case, the Koszul complex can be extended to the Koszul-Tate resolution by adding the polynomial algebra generated by 'anti-ghosts of anti-ghosts' (given degree -2), etc. To preserve the crucial Poisson algebra structure, one also adds 'ghosts of ghosts' (given degree 2), etc.

In general, the quotient space is not a manifold, often not even Hausdorff, in which case $H^0(\mathcal{BFV})$ provides a suitable candidate for the algebra of observables on the 'reduced phase space'.

Since $\mathcal{BFV}$ is a free graded commutative algebra over A, assuming sufficient finiteness, the differential derivation $d_{\mathcal{BFV}}$ is graded dual to a differential coderivation on a free graded cocommutative coalgebra over A and hence is equivalent to an L_∞-algebra. This is spelled out in considerable detail by Kjeseth [Kje01a, Kje01b].

6 Lagrangians with symmetries

Lagrangian physics derives 'equations of motion' from a variational principle of least action. Here an action refers to an integral

$$S(\phi) = \int_M L((j^n \phi)(x)) vol_M$$

over some manifold M where ϕ is a (possibly vector valued) function on M or section of a bundle E over M. The action may have symmetries, i.e. variations in ϕ which do not change the value of S and hence are physically irrelevant in the sense that ϕ and its transformed value encode the same physical information.

Emma Noether had two major theorems regarding the variational calculus. The first, much better known and often referred to as *Noether's theorem*, asserts a correspondence between symmetries and conserved quantities. Noether's second variational theorem establishes a correspondence between symmetries, notably gauge symmetries, and differential algebraic relations among the Euler-Lagrange equations. It is this second theorem that has an important role in the Batalin-Vilkovisky construction for Lagrangians with symmetries.

These symmetries create difficulties for quantization of such physical theories. The method of Batalin and Vilkovisky [BV84, BV83] was invented to handle these difficulties, but turns out to be of interest also in a classical context. The construction is quite parallel to that of Batalin-Fradkin-Vilkovisky in the constrained Hamiltonian case, but with one crucial difference: instead of a grading preserving bracket, they use an 'anti-bracket' (independently due to Zinn-Justin [ZJ75, ZJ76]) which is of degree 1. Therefore it is also known as an odd Poisson or Gerstenhaber bracket. In this Lagrangian setting, Batalin and Vilkovisky extend the BRST cohomological approach by introducing anti-fields (independently and previously due to Zinn-Justin) dual to the original fields and anti-ghosts which (with hindsight) correspond to the Noether relations and are dual to the ghosts which generate the BRST complex for the Lie algebra of symmetries.

The relevance of Noether's theorem is not emphasized in most of the literature using the BV approach. As with $\mathcal{BFV}$, part of the differential of the Batalin-Vilkovisky complex $\mathcal{BV}$ is that of the Koszul-Tate resolution, in this case of the differential ideal generated by the Euler-Lagrange equations. The anti-fields generate the Koszul complex, which is not a resolution; the anti-ghosts provide the next level of generators, as described by Tate [Tat57], corresponding to the relations among the Euler-Lagrange equations. It is the full acyclicity of the Koszul-Tate resolution that permits the application of Homological Perturbation Theory and thus guarantees the existence of the terms of higher order in the full differential $d_{\mathcal{BV}}$. As in the concluding remark in Section 5, the graded dual to $d_{\mathcal{BV}}$ is equivalent to an L_∞-algebra. We comment on this further in Section 6.1.

Rather than carrying out this analysis in the abstract, we mention two particularly striking realizations of this structure: the Poisson sigma models of Cattaneo and Felder [CF99] and our analysis with Fulp and Lada [FLS02a, FLS02b] of Lagrangians with field dependent symmetries as in the case of higher spin particles.

6.1 Field dependent gauge symmetries

Field dependent gauge symmetries appear in several field theories, most notably in a class due to Ikeda [Ike94] and Schaller and Strobl [SS94], including the Poisson sigma model of Cattaneo and Felder [CF99] above. A significant generalization occurs in the Berends, Burgers and van Dam [Bur85, BBvD86, BBvD85] approach to "particles of spin ≥ 2". The physics of "particles of spin ≤ 2" leads to representations of a Lie algebra Ξ of gauge parameters on a vector space Φ of fields. By a field dependent action of Ξ on Φ, Berends, Burgers and van Dam mean a polynomial (or power series) map $\delta(\xi)(\phi) = \Sigma_{i\geq 0} T_i(\xi, \phi)$ where T_i is linear in ξ and polynomial of homogeneous degree i in ϕ. Berends, Burgers and van Dam consider arbitrary field theories, subject only to the requirement that the commutator of two gauge symmetries be another gauge symmetry whose gauge parameter is possibly field dependent. Thus they do not require an a priori given Lie structure to induce the algebraic structure of the gauge symmetry "algebra".

Let Φ denote the vector space of fields and Ξ the vector space of gauge parameters. Let $\Lambda^*\Phi$ denote the free graded cocommutative coalgebra cogenerated by Φ. Although the space Ξ of gauge parameters has no natural Lie structure, the space of linear maps from $\Lambda^*\Phi$ into Ξ is a Lie algebra under certain mild assumptions along with a hypothesis which we refer to as the *BBvD hypothesis*. Under these assumptions, the gauge algebra gives rise to an L_∞-algebra on a differential graded vector space V with Ξ in degree 0, Φ in degree 1 and 0 in all other degrees. Take $\partial : \Xi \to \Phi$, given by $\partial(\xi) = \delta(\xi)(1) \in \Phi$, as the only non-trivial differential. Define

$$D : \Lambda^*(sV) \to sV$$
$$D(\xi) = \partial(\xi)$$
$$D(\xi \wedge \phi_1 \wedge \cdots \wedge \phi_n) = \delta(\xi)(\phi_1 \wedge \cdots \wedge \phi_n) \text{ for } n \geq 1$$
$$D(\xi_1 \wedge \xi_2 \wedge \phi_1 \wedge \cdots \wedge \phi_n) = C(\xi_1, \xi_2)(\phi_1 \wedge \cdots \wedge \phi_n)$$

and $D = 0$ on elements of $\Lambda^*(sV)$ with more than two entries from Ξ or with no entry from Ξ.

Notice this is essentially *not* of the same form as that of Roytenberg and Weinstein in section 4.1, although both have just two components. The crucial difference is in the grading: $0, 1$ here versus $-1, 0$ for them.

Theorem 6.1 *$D : \Lambda^*(sV) \to sV$ as defined above gives V the structure of an L_∞-algebra*

7 L_∞-maps, deformation quantization, String Field Theory (SFT) and more

Definition 7.1 *An L_∞-map $f : \mathfrak{h} \to \mathfrak{g}$ of L_∞-algebras (or dg Lie algebras) is a dg coalgebra map $\Lambda^c(s\mathfrak{h}) \to \Lambda^c(s\mathfrak{g})$.*

The Cattaneo and Felder Poisson sigma model was developed to provide an alternative, 'path integral', proof of Kontsevich's theroem that any Poisson *manifold* can be deformation quantized. In both proofs, the key issue is the *formality* of a certain dg Lie algebra $\mathfrak{g}$. The L_∞-equivalence of this $\mathfrak{g}$ and $H(\mathfrak{g})$ implies that all the obstructions to deformation quantization vanish.

For this important application, Σ was a disk so the maps $\Sigma \to M$ could be considered as world sheets as in SFT.

The relevance of A_∞- and L_∞-structure to (respectively) OSFT and CSFT has a particularly 'physical' interpretation. The higher order operations describe multiple string interactions, *not* obtained from 3-string interactions (multiplication, respectively bracketing of 2 strings) or the equivalent *correlation* functions [Zwi93]. Here too there is contact with Alan and his student Tang in their recent paper.

Because BBvD give an explicit expansion T_i, C_i, the corresponding multibrackets l_i are visible or at least easy to extract. In contrast, in the Cattaneo and Felder Poisson sigma model, they are hidden in the single *function* α and it's derivatives.

8 Homological Legendre transform

In their concluding remarks in [RW98], Dmitry and Alan muse:

> L_∞-algebras occur in physics in the framework of the Batalin-Vilkovisky procedure for quantizing gauge theories. On the other hand, the Courant bracket seems to provide a geometric framework for constrained Hamiltonian systems. It is known [HT92] that gauge Lagrangians lead to constrained theories in the Hamiltonian formalism. This suggests that homotopy Lie algebras arising in the Batalin-Vilkovisky formalism and those in the Courant formalism might be somehow related.

In response to my paper for the Alanfestschrift, Dmitry pointed out to me the paper of Grigoriev and Damgaard [GD00] which establishes an analog of the Legendre transform in terms of the BFV and BV constructions. Here at Alanfest, we have investigated their transforms in further detail. This bears further investigation, but for now let me mention only that in either direction, Hamiltonian to lagrangian of vice versa, the essential idea is the 'oddification' of all the fields, ghosts, etc., then substituting these into the respective formulas for the Hamiltonian, the Lagrangian, etc. and keeping only the parts of the appropriate total degree. Dmitry can interpret this further by looking at the algebra as that of a graded path space.

9 Coda

There are still other examples of A_∞- and L_∞-structures with potential physical relevance, for example, Fukaya's A_∞-categories, but that takes us further afield from today's topic: the L_∞-structures directly involved in some of Alan's work and closely related to his foundational work on symplectic reduction. There are further relations to be discovered, as he has indicated. Leaving that for the future, let me conclude with best wishes for the continuation of a long happy and inspiring career to Alan on his 60th birthday.

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